Hilroy

# CALCULUS II

MATH 138

**80** Pages
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**EXERCISE BOOK**
**CAHIER D'EXERCICES**

**NAME/NOM** ______________________

**SUBJECT/SUJET** ______________________

ASSEMBLED IN CANADA WITH IMPORTED MATERIALS
ASSEMBLÉ AU CANADA AVEC DES MATIÈRES IMPORTÉES

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# Intro- Integration

-> Breaking things into chunks

-> Ex. $[a,b]$ -> $a \overset{t_0}{\bullet}\!-\!\overset{t_1}{|}\!-\!\overset{t_2}{|}\!-\!-\!\overset{t_3}{|}\overset{t_4}{|} \cdots \overset{t_n}{\bullet} b$

-> The increasing sequence $t_n$ is a partition of $[a,b]$

-> The length of a given subinterval is: $\Delta t_i = t_i - t_{i-1}$, $i \in \{1, 2 \ldots n\}$

-> Now, let $c_i \in [t_{i-1}, t_i]$ ($c_i$ is some point in the interval

$a \bullet \overset{c_1}{\times} | \overset{c_2}{\times} | \overset{c_3}{\times} \; | \overset{c_4}{\times} | \overset{c_5}{\times} \bullet b$

Riemann Sums

-> Defn: Riemann Sum: Given a bounded function $f$ and partition $P$ over the interval $[a,b]$ with $c_i \in [t_{i-1}, t_i]$, a Riemann sum of $f$ with respect to $P$ is

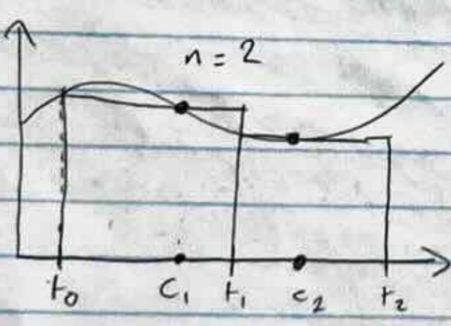
$$S = S(f,P) = \sum_{i=1}^{n} f(c_i)\Delta t_i = f(c_1)\Delta t_1 + f(c_2)\Delta t_2 + \cdots$$

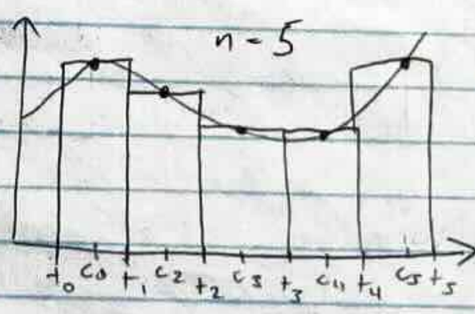
$\Delta x = \frac{b-a}{n}$

Right sum: $x_i = x_0 + i\Delta x$

-> Different partitions and choices for $c_i$ will yeild different values of $S$

-> The value of $n$ (# of subintervals) can be different for different Riemann sums




# Increasing Accuracy

-> As $n$ gets bigger, the area under the curve gets better approximated

or rather $\lim_{n \to \infty}$ -> If we let $n = \infty$, each term in the sum is 0, so we end up with $0 \cdot \infty$ (indeterminate)

-> If $n = \infty$, the way $[a,b]$ was partitioned (and the choices of c) don't matter: sum = the same

-> If that last property holds for a function, it is integrable

-> We define $\|P\|$ (norm of P) as $\max\{\Delta t_1, \Delta t_2 \ldots\}$, we can state our idea as Integrable: $\|P\| \to 0$ as $n \to \infty$

-> Regular n-partition of $[a,b]$: partition with $n$ parts that are all the same size

## Integrals

-> If our bonded function f is integrable with the value I, we have $I = \int_b^a f(t)\,dt$ ← variable of integration

limits of integration ↑ integrand

-> t is a dummy variable: the letter doesn't matter
  -> t does not turn up in the final answer (like an index in a for loop)

-> Ex. calculate the integral of $e^x$ over $[1,4]$:

$$\int_1^4 e^x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{c_i} \Delta t_i\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} e^{c_i} \frac{3}{n}\,dt =$$

We can use a right hand Riemann sum (defining $c_i$ as the right endpoint of $[t_{i-1}, t_i]$:

$$= \lim_{n\to\infty} \sum_{i=1}^{n} e^{1+\frac{3}{n}} \left(\frac{3}{n}\right) = \lim_{n\to\infty} \frac{3e}{n} \sum_{i=1}^{n} [e^{\frac{3}{n}}]^i$$

$\boxed{\text{geometric sum: } \sum_{i=1}^{n} r^i = \frac{r^{n+1} - r}{r-1}}$

$$= \lim_{n\to\infty} \frac{3e}{n} \left(\frac{e^{\frac{3}{n}+3} - e^{\frac{3}{n}}}{e^{3/n} - 1}\right)$$

$\boxed{\text{and now, we have a limit we can solve normally!}}$

$$= \frac{3e(e^3-1)}{3} = e(e^3-1)$$

-> But is $e^x$ integrable? yes!
  -> if f is continuous on $[a,b]$, it is integrable on $[a,b]$
  -> If f has finitely many jump discontinuities, it will also be integrable

## Properties of Integrals

-> If f and g are integrable over $[a,b]$, then

(a) $\int_b^a c \cdot f(t)\,dt = c \cdot \int_b^a f(t)\,dt$ (constants can be factored)

(b) $\int_b^a [f(t) + g(t)]\,dt = \int_b^a f(t)\,dt = \int_b^a g(t)\,dt$

(c) If $m \le f(t) \le M$, then $m(b-a) \le \int_b^a f(t)\,dt \le M(b-a)$

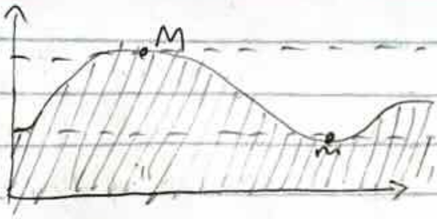
(d) $|f|$ is integrable on $[a,b]$. $\left|\int_b^a f(t)\,dt\right| \le \int_b^a |f(t)|\,dt$

## Proof of ⓒ

→ Note that for any partition $P$, note that $\sum_{i=1}^{n} \Delta t_i = b-a$

Since $m \le f(t) \le M$, we get

$$\sum_{i=1}^{m} m \, \Delta t_i \le \sum_{i=1}^{n} f(t) \Delta t_i \le \sum_{i=1}^{n} M \Delta t_i$$




This is true for any partition $P$, so we end up with $m(b-a) \le \int_a^b f(t)\,dt \le M(b-a)$

## Corollaries

ⓔ if $m=0$ (from ⓒ), if $f(t) \ge 0$, then $\int_a^b f(t)\,dt \ge 0$

ⓕ if $f(t) \ge g(t)$, then $\int_a^b f(t)\,dt \ge \int_a^b g(t)\,dt$

## Extra properties

→ These do not follow directly from the sum def, but are rather defined

→ We define that $\int_a^a f(t)\,dt = 0$ (any riemann sum would multiply 0 by a finite number (a)

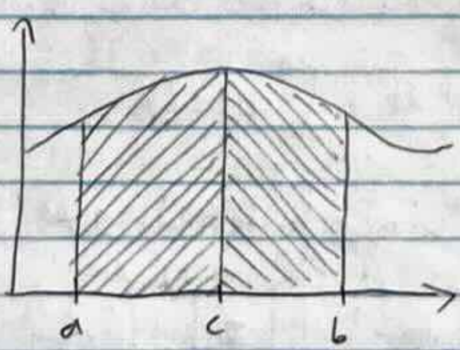
→ We have $\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$

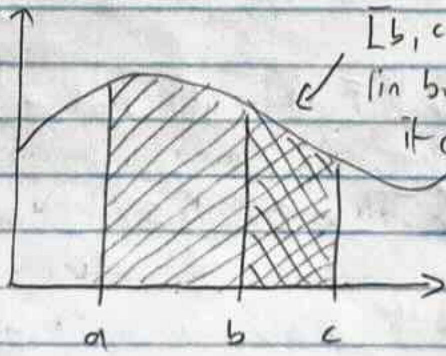
## "Breaking up" Theorem

→ Given $a,b,c \in I$ over which $f$ is integrable, then

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$$

→ Note: it is not required that $a \le c \le b$ or $b \le c \le a$



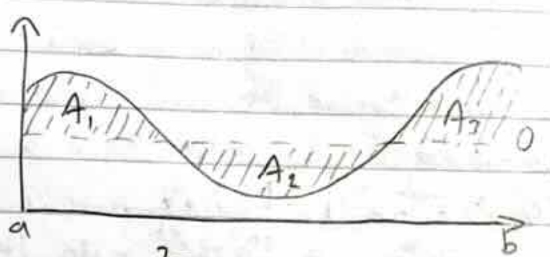




[b, c] is covered twice (in both directions), so it cancels out

## Areas and Integrals

-> $\int_a^b f(t)\,dt$ will returned the signed area under the curve of f, not the expected area

-> That is, if $f < 0$ over some interval $[c, d]$ then $\int_a^b f(t)\,dt$ will return the negative area between f and the x-axis




-> here, $A_1$ and $A_3$ are positive and $A_2$ is negative, so $\int_a^b f(t)\,dt = A_1 + A_3 - A_2$

-> Ex. $\int_0^{2\pi} \sin(t)\,dt = 0$ since equal areas are above and below the x-axis

## Average Values

-> Recall, the average of the discrete set $\{x_1, x_2 \ldots\}$ is $\sum_{i=1}^{n} x_i / n$

-> The average value of a continuous function over the interval $[a, b]$ is given by

$$f_{av} = \bar{f} = \frac{1}{b-a}\int_a^b f(t)\,dt$$

-> Geometric hint: dividing the area under the curve by the length of the interval

-> Ex. Compute $\bar{f}$ over $[0, 4]$ if $f(x) = 3x$ geometrically

-> Since f is positive and forms a triangle with the x and y axes, we can compute the area as

$$\int_0^4 3x\,dx = \frac{bh}{2} = \frac{4 \cdot 12}{2} = 24 \longrightarrow$$

$$\bar{f} = 24/(4-0) = \frac{24}{4} = 6$$

-> Here, $\bar{f}_{av}$ occurs halfway between 0 and 12

-> In all cases, $\bar{f}$ will split $f(x)$ into 2 parts of equal area

-> The area below $f(x)$ and above $\bar{f}$ will equal the area above $f(x)$ and below $\bar{f}$

-> This can be proven by shifting the function so that $\overline{f}$ becomes the x-axis. let $g(x) = f(x) - \overline{f}$

-> If the areas above and below $\overline{f}$ are equal, $\int_a^b g(t)dt$ will be equal to 0 (since the "area" above is positive and below is negative).

-> Is it always the case that for an integrable function f, there exists a $c \in [a,b]$ such that $f(c) = \overline{f}$

-> In general, NO, but it can be made YES with some restrictions

-> Average Value Theorem (Mean Value Theorem for Integrals): If f is continuous on $[a,b]$ then there is a $c \in [a,b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

-> Proof: By EVT, there exist $p, q \in [a,b]$ such that $f(p) \leq f(x) \leq f(q)$ for all $x \in [a,b]$. By integral properties:

$$(b-a)f(p) \leq \int_a^b f(x)\,dx \leq (b-a)f(q) \Rightarrow$$

$$f(p) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq f(q)$$

Since f is continuous, by the IVT there is a $c \in [a,b]$ where $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = \overline{f}$

## Fundamental Theorem of Calculus (FTC)

-> Until now, to calculate integrals, we have had to rely on geometry or converting the limits of riemann sums (the sum must be converted to an explicit expression)

-> In many cases, it is simply impossible to do this

-> Let us consider a function $A(x)$ where x is a parameter of an integral expression: $A(x) = \int_a^x f(t)\,dt$

-> Now consider x+h. $A(x+h) = \int_a^{x+h} f(t)\,dt$ would be a slightly larger interval, so the area under it $A(x)$ would change slightly as well

-> The incremental area is given by $A(x+h) - A(x)$

properties of integrals

$$= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

-> Divide both sides by h. This gives

$$\frac{A(x+h) - A(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt \quad \text{looks familiar?}$$

-> Since f is continuous, by AVT, there exists $c \in [x, x+h]$ such that $(A(x+h) - A(x))/h = f(c)$

-> Also since f is continuous, $\lim_{h \to 0} f(c) = f(x)$

-> Finally, by definition: $\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = A'(x)$

-> Thus, $A'(x) = f(x)$ , $\frac{d}{dx} \int_a^x f(t)dt = f(x)$

-> That is to say, integration cancels differentiation

Theorem: FTC I

-> If f is continuous on an open interval containing x and a, then $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$ ~~That is to say, the derivative~~ cancels the integral

How is this helpful?

-> If we let $g(x) = \int_a^x f(t)\,dt$, by FTC I we know that $g'(x) = f(x)$, or rather $f(x)$ is the anti-derivative of $g(x)$

-> Ex. Compute $\int_3^5 2t\,dt$

-> We need a function $g(x)$ such that $g'(x) = 2x$; we need the antiderivative of $2x$.

-> Let $F(x) = x^2 + C$. This is the family of antiderivatives of $2x$.

-> By FTC I we have $G(x) = x^2 + C = \int_3^x 2t(dt)$

-> To compute $\int_3^5 2t\,dt$, we compute $G(5) = 25 + C$

-> We must get rid of C

-> We know $G(3) = \int_3^3 2t\,dt = 0 = 9 + C$, so $C = -9$

-> So $G(5) = 25 + (-9) = 16$

-> Notice that the constant always gets cancelled out

Theorem: FTC II

-> Let F be any antiderivative of a continuous function f. Then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

→ Ex Compute $\int_1^4 \cos(x)\,dx$

→ Using a Riemann sum would require a formula for $\sum_{i=1}^{n} \cos(1+\frac{3i}{n})$ and a limit $n \to \infty$

→ Possible, but very hard

→ Using FTC, since $\frac{d}{dx}\sin(x) = \cos(x)$, we have

$$\int_1^4 \cos(x)\,dx = \sin(4) - \sin(1)$$

→ Notation: $g(x)\Big|_a^b = g(b) - g(a)$

→ Ex. compute $\int_{-1}^{1} |x|\,dx$

→ For stepwise functions, split the integral: $\int_{-1}^{1} |x|\,dx = \int_{-1}^{0} -x\,dx + \int_0^1 x\,dx$

$$= -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_0^1 = 0-(-\frac{1}{2}) + \frac{1}{2} - 0 = \frac{1}{2} + \frac{1}{2} = 1$$

## Extended FTC II

→ We learned that $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

→ What about $\frac{d}{dx}\int_x^a f(t)\,dt$ or $\frac{d}{dx}\int_{\ln(x)}^{x^3} f(t)\,dt$ ?

→ Ex. compute $\int_2^{\ln(x)} f(t^2)\,dt$

→ Let $u(x) = \ln(x)$ and let $g(u(x)) = \int_2^u \sin(t^2)\,dx$

→ By the chain rule, $\frac{d}{dx} g(u(x)) = \frac{d}{du} g(u(x)) \frac{d}{dx} u(x)$

→ Now $\frac{du}{dx} = \frac{1}{x}$ and by FTC $\frac{d}{dx} g(u) = \frac{d}{du}\int_2^u \sin(t^2) = \sin(u^2)$

→ so $\frac{d}{dx} g(u) = \sin(u^2)(\frac{1}{x}) = \sin(\ln(x)^2)(\frac{1}{x})$

## Chain Rule for Integration

→ For a continuous and differentiable $a(x)$ and $b(x)$, we get

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)$$

→ Tip: this is simple the derivative of FTC II

$$\int_{a(x)}^{b(x)} f(t)\,dt = F(b(x)) - F(a(x))$$

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = F'(b(x))b'(x) - F'(a(x))a'(x)$$

$$= f(b(x))b'(x) = f(a(x))a'(x)$$

## Indefinite Integral

-> So far, we have defined $\int_a^b f(t)\,dt$ as the definite integral of $f$ over $[a,b]$. It evaluates to a number

-> We call $\int f(x)\,dx$ the indefinite integral of $f$. It is a function

-> Due to FTC, it represents the family of antiderivatives of $f(x)$

-> ex. $\int 4x+3\,dx = 2x^2+3x+C$

## Basic antiderivatives

-> $\int x^n\,dx = \frac{1}{n+1}x^{n+1}+C,\ n \neq -1$ (reverse power rule)

-> $\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + C$

-> $\int e^x = e^x + C$ -> $\int a^x\,dx = \frac{1}{\ln(a)}a^x + C$ $(a>0,\ a\neq 1)$

-> $\int \sin(x)\,dx = -\cos(x)+C$ -> $\int \cos(x)\,dx = \sin(x)+C$

-> $\int \sec^2(x)\,dx = \tan(x)+C$

-> $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)+C$ -> $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x)+C$

-> Ex. compute $\int 2x+\sin(x)-1\,dx$

-> By linearity of integration, we have $\int 2x\,dx + \int \sin(x)\,dx + \int dx$

$= x^2-\cos(x)-x+C$

## Antiderivatives are hard

-> It can be very hard to find antiderivatives

-> Sometimes impossible: $e^{x^2}$, $\sqrt{1+x^3}$, $\ln(\ln(x))$, $\sin(x^2)$

-> When they do exist, finding them is not as formulaic as finding derivatives (relevant xkcd)

-> We have a number of strategies that sometimes help

## Substitution Rule (u-substitution)

-> First we note that by FTC, $\int g'(x)\,dx = \int \frac{d}{dx} g(x)\,dx = g(x)+C$

-> This is sometimes written as $\int d[g(x)] = g(x)+C$

-> Consider $\int \sin^3(x)\cdot\cos(x)\,dx$

 -> Notice that $\cos(x)$ is the derivative of $\sin(x)$

-> Let $u=\sin(x)$. Then $\frac{d}{dx}u = \cos(x)$

-> This gives us $\int u^3 \cdot \frac{du}{dx}\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4(x)+C$

 -> We use FTC and the chain rule here

 -> Here, we "cancel" the dx like a fraction. There is a more complicated (and mathematically justified) way to calculate using u-sub

-> SUMMARY: When we identify $\int f(g(x))\,g'(x)$, we can let $u=g(x)$ to get $\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$

 -> We can manipulate du and dx algebraicly

 -> Ex. Consider $\int \sin^3(x)\cos(x)\,dx$. Solve using u-sub

  -> Let $u=\sin(x)$ so that $\frac{du}{dx}=\cos(x) \Rightarrow dx = \frac{du}{\cos(x)}$

  which gives $\int u^3\cos(x)\frac{du}{\cos(x)} = \int u^3\,du = \frac{1}{4}u^4+C = \frac{1}{4}\sin^4(x)+C$

-> At the end, check the integral by taking the derivative

## Extra Substitution Rule Stuff

-> Ex. Compute $\int 5x\,e^{x^2}\,dx$

 -> Let $u=x^2$, then $\frac{du}{dx}=2x \Rightarrow dx = \frac{du}{2x}$. This gives

 $$\int 5xe^u\frac{du}{2x} = \int \frac{5}{2}e^u\,du = \frac{5}{2}\int e^u\,du = \frac{5}{2}e^u+C = \frac{5}{2}e^{x^2}+C$$

-> There is no set rule for choosing u: just look for a function where it and its derivative appear in the integrand

 -> Ex. $\sin^3(x)\cos(x)$, $5xe^{x^2}$, $\frac{\ln x}{x}$

 -> However, this doesn't always work

 -> Ex. $\int x^3\sqrt{x^2-4}\,dx$. Since $\frac{d}{dx}x^3 = 3x^2$, let $u=x^3$. We have

 $$\int x^3\sqrt{x^2-4}\,dx = \int u\sqrt{x^2-4}\,dx = \int u\sqrt{x^2-4}\,\frac{du}{3x^2}$$

 $= \frac{1}{3}\int \frac{u}{x^2}\sqrt{x^2-4}\,du$. Since $u=x^3$, $x^2=u^{2/3}$. We get

 $\frac{1}{3}\int \frac{u}{u^{2/3}}\sqrt{u^{2/3}-4}\,du = \frac{1}{3}\int u^{\frac{1}{3}}\sqrt{u^{2/3}-4}\,du$ ... bruh

 -> Nothing really changed

-> Ideally, we want to end up with a basic antiderivative formula like $\int x^n\,dx$ or $\int \sin(x)\,dx$, so we should try something else

-> In this example, letting $u = x^2-4$ gives a better result

-> $u = x^2-4 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$$\int x^3\sqrt{u}\,\frac{du}{2x} = \int \frac{x^2}{2}\sqrt{u}\,du = \frac{1}{2}\int (u+4)\sqrt{u}\,du = \frac{1}{2}\int u^{\frac{3}{2}} + 4u^{1/2}$$

$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} + (4)\frac{2}{3}u^{3/2}\right) + C = \frac{1}{5}u^{5/2} + \frac{4}{3}u^{3/2}$$

$$= \frac{(x^2-4)^{5/2}}{5} + \frac{4}{3}(x^2-4)^{3/2} + C$$

-> Ex. Solve $\int e^{5x}\,dx$

-> let $u = 5x \rightarrow \frac{du}{dx} = 5 \rightarrow dx = \frac{du}{5}$

$\rightarrow \int e^u \frac{du}{5} \rightarrow \frac{1}{5}\int e^u\,du \rightarrow \frac{1}{5}e^u + C \rightarrow \frac{e^{5x}}{5} + C$

-> We can often eyeball the coefficient required in the denominator as long as it is a constant

## Substitution with definite Integrals

-> It is important to realize $\int_a^b f(x)\,dx$ means $x=a$ and $x=b$

-> That is, $\int_a^b f(x)\,dx$ can be written as $\int_{x=a}^{x=b} f(x)\,dx$

-> If we transform a variable $u(x)$, we need to adjust the limits (bounds) of integration

$\rightarrow \int_a^b f(g(x))g'(x)\,dx \neq \int_a^b f(u)\,du$. Instead, we have

$\rightarrow \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,dx$

-> Ex. $\int_1^e \frac{1}{x(2+\ln(x))}\,dx$

-> Option one: let $u = 2+\ln(x)$, then $\frac{du}{dx} = \frac{1}{x} \Rightarrow x\,du = dx$

-> Since we switch from $dx$ to $du$, we must switch our bounds:

$x=1 \Rightarrow u=2$, $x=e \Rightarrow u=3$. These are the new bounds

$$\int_1^e \frac{1}{x(2+\ln(x))}\,dx = \int_2^3 \frac{1}{ux}x\,du = \int_2^3 \frac{1}{u}\,du = \ln|u|\Big|_2^3 = \ln(3)-\ln(2)$$

$$= \ln\left(\frac{3}{2}\right)$$

-> Option two: Find a general antiderivative first, then solve that using FTC

-> let $u = 2 + \ln(x) \Rightarrow \dots \Rightarrow x\,du = dx$

-> $\int \frac{1}{x(2+\ln(x))}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|2+\ln(x)| + C$

$\therefore$ If $f(x) = \frac{1}{x(2+\ln(x))}$, then $F(x) = \ln|2+\ln(x)| + C$

So $\int_1^e \frac{1}{x(2+\ln(x))}dx = \ln|2+\ln(x)|\Big|_1^e = \ln|2+1| - \ln|2| = \ln\left(\frac{3}{2}\right)$

-> Practice exercise: $\int_0^1 x(x-1)^6(x+1)^6\,dx$ (ans $= \frac{1}{14}$)

-> Ex. Calculate $\int \frac{1}{\sqrt{1+\sqrt{1+x}}}dx$

-> let $u = \sqrt{1+x}$, $\frac{du}{dx} = \frac{1}{2\sqrt{1+x}} \Rightarrow du = \frac{1}{2\sqrt{1+x}} = \frac{1}{2u}dx$

$= 2u\,du$

-> So we get $\int \frac{1}{\sqrt{1+\sqrt{1+x}}}dx = \int \frac{2u}{\sqrt{1+u}}du$

-> Now let $s = 1+u$ (we are making a second substitution)

-> $ds = du$, so we have

$\int \frac{2u}{\sqrt{1+u}}du = \int \frac{2(s-1)}{\sqrt{s}}ds = 2\int \sqrt{s} - \frac{1}{\sqrt{s}}\,ds = 2\left[\frac{2}{3}s^{3/2} - 2s^{1/2}\right] + C$

$= \frac{4}{3}s^{3/2} - 4s^{1/2} = \frac{4}{3}(u+1)^{3/2} - 4(u+1)^{1/2} = \frac{4}{3}(1+\sqrt{1+x})^{3/2}$
$- 4(1+\sqrt{1+x}) + C$

## Inverse Trig Sub

-> Recall

-> $1 - \sin^2\theta = \cos^2\theta$, $\tan^2\theta + 1 = \sec^2\theta$, $\sec^2\theta - 1 = \tan^2\theta$

-> When an integrand involves an expression $a^2 - x^2$, $x^2 + a^2$ or $x^2 - a^2$, then we may make the following substitutions

for $a^2 - x^2 \to x = a\sin\theta$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

for $a^2 + x^2 \to x = a\tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

for $x^2 - a^2 \to x = a\sec\theta$, $\theta \in \left[-\pi, -\frac{\pi}{2}\right) \cup \left[0, \frac{\pi}{2}\right)$

-> We usually do this for things like $(a^2 - x^2)^p$, $(x^2+a^2)^p$, $(x^2-a^2)^p$ where $p$ is a fraction (otherwise we can just expand)

# Trig Sub Examples

→ Consider the integral $\int \frac{x}{\sqrt{9+x^2}}\,dx$ (we didn't think of u-sub)

→ → We will sub $x \to 3\tan\theta$. $dx = 3\sec^2\theta\,d\theta$

→ We have $\int \frac{3\tan\theta}{\sqrt{9+9\tan^2\theta}} 3\sec^2\theta\,d\theta = \int 3\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta$

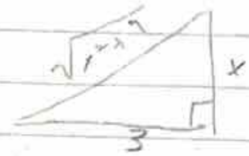
$= 3\int \frac{\tan\theta\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta = 3\int \frac{\tan\theta\sec^2\theta}{|\sec\theta|}\,d\theta = 3\int\frac{\tan\theta\sec^2\theta}{\sec\theta}\,d\theta$

use trig ratios!

(since $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$) $= 3\int \sec\theta\tan\theta\,d\theta = 3\sec\theta + C$

→ what is $\sec\theta$ in terms of $x$?

→ $x = 3\tan\theta \Rightarrow \frac{x}{3} = \tan\theta$ →




→ so $\sec\theta = \sqrt{x^2+9}/3$

→ so ... $= 3\left(\frac{\sqrt{x^2+9}}{3}\right) + C = \sqrt{x^2+9} + C$

→ Ex. $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

→ let $x = \sin\theta$. We have $dx = \cos\theta\,d\theta$ with $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

→ We get $\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \int\frac{\sin^2\theta}{\sqrt{\cos^2\theta}}\cos\theta\,d\theta$

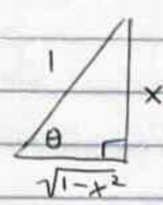
$= \int \frac{\sin^2\theta}{|\cos\theta|}\cos\theta\,d\theta = \int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta$ (since $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$)

$= \int \sin^2\theta\,d\theta = \int \underbrace{\frac{1-\cos 2\theta}{2}}_{\text{identity}}\,d\theta = \frac{1}{2}\int 1-\cos 2\theta\,d\theta$

identity →

$= \frac{1}{2}\left(\int 1 - \int\cos 2\theta\,d\theta\right) = \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C = \frac{1}{2}\left[\theta - \sin\theta\cos\theta\right]$

We have $x = \sin\theta$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so $\theta = \sin^{-1}(x)$




→ $\cos\theta$ will be $\cos(\arcsin(x)) = \sqrt{1-x^2}$

$\therefore\ = \frac{1}{2}\left(\sin^{-1}(x) - x\sqrt{1-x^2}\right) + C$

→ Aside: Even powers of trig functions often require identities

→ Aside 2: Odd powers of sin and cos can be done as follows

$\int \cos^5 x\,dx = \int \cos^4 x\,\cdot$

## Trig Sub Examples (cont.)

-> Consider $\int 1/(x^2+4x)^{3/2}\,dx$ (general quadratic)

-> we must complete the square to do this:

$$\int \frac{1}{((x+2)^2-4)^{3/2}}\,dx$$ Now let $x+2 = 2\sec\theta$ ($\theta \in$ Quadrant 3 or 1)

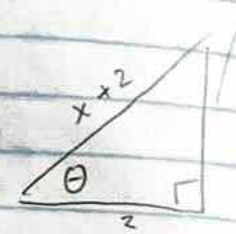
-> $dx = 2\sec\theta\tan\theta$

$$= \int \frac{1}{[4\sec^2\theta - 4]^{3/2}} \cdot 2\sec\theta\tan\theta\,d\theta = \frac{1}{4}\int \frac{\sec\theta\tan\theta}{(\sec^2\theta-1)^{3/2}}\,d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta\tan\theta}{(\tan^2\theta)^{3/2}} = \frac{1}{4}\int \frac{\sec\theta\tan\theta}{|\tan\theta|^3} = \frac{1}{4}\int\frac{\sec\theta\tan\theta}{\tan^3\theta}$$ (since $\theta \in$ quad 1 or 3)

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{4}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta.$$ Let $u = \sin\theta$, $du = \cos\theta\,d\theta$

$$= \frac{1}{4}\int \frac{1}{u^2}\,du = \frac{1}{4}\left(-\frac{1}{u}\right) + C = \frac{1}{4\sin\theta} + C = -\frac{\csc\theta}{4} + C$$




Since $x+2 = 2\sec\theta$, we have $-\dfrac{x+2}{4\sqrt{(x+2)^2-4}} + C$

## Integration by Parts

-> Recall the product rule: $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$

-> We integrate both sides to get $f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$

-> Re-write: $\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)$

-> Finally, we have $\boxed{\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx}$

-> It is often easier to evaluate $\int F(x)\,d(x)\,dx$ than $\int f(x)g(x)\,dx$

-> Be careful: $\int f(x)g(x)\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx$

-> IPB is also written $\int u\,dv = uv - \int v\,du$

-> $u = g$, $du = g'dx$, $dv = f\,dx$ $v = F$

## Integration by Parts Examples

-> Find $\int xe^{2x}\,dx$ [here: $g(x) = x$ and $f(x) = e^{2x}$]

-> $\int xe^{2x}\,dx = \frac{1}{2}e^{2x}x - \int \frac{1}{2}e^{2x}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$

-> Find $\int x\sin(x)\,dx$ [$g(x) = x$, $f(x) = \sin(x)$]

-> $\int x\sin(x) = -\cos(x)x - \int -\cos(x)\,dx = \int \cos(x)\,dx - x\cos(x)$

$= \sin(x) - x\cos(x) + C$

-> Find $\int \ln(x)\,dx$ [one function, so $f(x) = 1$, $g(x) = \ln(x)$]

-> $\int \ln(x)\,dx = x\ln(x) - \int x\frac{1}{x}\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + C$

## IBP Special Cases

-> We might need to combine substitution and IBP to solve integrals

-> ex. $\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \int \frac{x}{x^2+1}\,dx$ (IBP)

-> $\int \frac{x}{x^2+1}\,dx$ : let $u = x^2+1$. $\frac{du}{dx} = 2x$ -> $dx = \frac{du}{2x}$

-> $\int \frac{x}{u}\left(\frac{du}{2x}\right) = \int \frac{1}{2}\frac{x}{ux}\,du = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|\,du$

$= \frac{1}{2}\ln|x^2+1| + C.$

-> So $\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \frac{1}{2}\ln|x^2+1| + C$

-> Sometimes we need to apply IBP multiple times

-> $\int e^x\cdot\sin(x) = e^x\sin(x) - \int e^x\cos(x) = e^x\sin(x) - \left(e^x\cos(x) - \int e^x(-\sin(x))\,dx\right)$

$= e^x\sin(x) - e^x\cos(x) - \boxed{\int e^x\sin(x)\,dx} \Rightarrow 2\int e^x\sin(x)\,dx = e^x\sin(x) - e^x\cos(x)$

-> $\int e^x\sin(x) = \frac{1}{2}\left[e^x\sin(x) - e^x\cos(x)\right] + C$

## Integrating Rational Functions

→ We wish to integrate functions in the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x), q(x) \in \mathbb{R}[x]$ and $\deg(p(x)) < \deg(q(x))$

→ We start with some building blocks:

① Linear factors: $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$

Irreducible Quadratics

② Binomial Expansion: $\int \frac{1}{(ax+b)^n}\,dx = \frac{1}{a}\cdot\frac{1}{1-n}\cdot\frac{1}{(ax+b)^{n-1}} + C,\ n>1,\ n\in\mathbb{N}$

③ Irreducible, no middle term: recall: $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{x^2+1}$

→ Ex. $\int \frac{1}{2x^2+18}\,dx = \frac{1}{2}\int \frac{1}{x^2+9}\,dx = \frac{1}{18}\int \frac{1}{\frac{x^2}{9}+1}\,dx = \frac{1}{18}\int \frac{1}{(\frac{x}{3})^2+1}\,dx$

Let $u = x/3$, $dx = 3\,du$

$= \frac{1}{18}\int \frac{1}{u^2+1}\,3\,du = \frac{1}{6}\int \frac{1}{u^2+1}\,du = \frac{1}{6}\tan^{-1}(u) + C = \frac{1}{6}\tan^{-1}\left(\frac{x}{3}\right) + C$

No middle term, term on top

④ Ex. $\int \frac{3x}{x^2+5}\,dx$, let $u = x^2+5 \rightarrow \frac{du}{dx} = 2x \rightarrow dx = \frac{du}{2x}$

$\int \frac{3x}{u}\,\frac{du}{2x} = \frac{3}{2}\int \frac{1}{u}\,du = \frac{3}{2}\ln|u| + C = \frac{3}{2}\ln|x^2+5| + C$

⑤ With middle term: Ex. $\int \frac{1}{x^2-2x+5}\,dx$ [complete the square]

$\int \frac{1}{x^2-2x+5}\,dx = \int \frac{1}{(x-1)^2+4}$. Let $u = x-1$, $du = dx$ ⇒

$\int \frac{1}{u^2+4}$ (proceed to ③) $\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$

→ Example: $\int \frac{3x+1}{x^2-2x+5}\,dx$ : complete the square, split, follow ③, ④

$\rightarrow \int \frac{3x+1}{x^2-2x+5}\,dx = \int \frac{3x+1}{(x-1)^2+4} = \int \frac{3x}{(x-1)^2+4}\,dx + \int \frac{1}{(x-1)^2+4}\,dx$

$\ldots = \frac{3}{2}\ln\left|\left(\frac{x-1}{2}\right)^2+1\right| + 2\tan^{-1}\left(\frac{x-1}{2}\right) + C$

## Partial Fraction Decomposition

→ In order to use the building blocks ①–⑤, we must convert a general rational function $p(x)/q(x)$ into one of the forms

→ A proper rational function can be decomposed the simple rational functions with linear or irreducible quadratics in the denominator

→ Proper means $\deg(p(x)) < \deg(q(x))$

→ Ex. $\frac{8x+4}{x^4-1} = \frac{1}{x+1} - \frac{4x+2}{x^2+1} + \frac{3}{x-1}$

## PFD Steps

① Factor the denominator

→ $x^4-1 = (x^2+1)(x^2-1) = (x^2+1)(x+1)(x-1)$

② For each linear factor, construct a term of the form $\frac{A}{ax+b}$

→ A is unknown, a, b are known (and are from the factors in the denominators

→ $A/(x-1) + B/(x+1)$

For each irreducible quadratic factor, construct $\frac{Cx+D}{ax^2+bx+c}$.

→ $(Cx+D)/(x^2+1)$

③ If a factor is repeated n times, repeat step ② n times but with increasing exponents in the denominator

④ Solve for unknowns

$$\frac{8x+4}{x^4-1} = \frac{8x+4}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} \Rightarrow$$

We can write as

$$8x+4 = A(x+1)(x^2+1) + B(x^2+1)(x-1) + (Cx+D)(x-1)(x+1)$$

$$= A(x^3+x^2+x+1) + B(x^3-x^2+x-1) + Cx^3+Dx^2-Cx-D$$

$$= (A+B+C)x^3 + (A-B+D)x^2 + (A+B-C)x + (A-B-D)$$

$$0x^3 + 0x^2 + 8x + 4 =$$

$$= (A+B+C)x^3 + (A-B-D)x^2 + (A+B-C)x + (A-B-D)$$

→ This gives 4 sets of coefficients that must equal each other, so we get a system of 4 equations

$$\begin{aligned} A+B+C&=0 \\ A-B+D&=0 \\ A+B-C&=8 \\ A-B-D&=4 \end{aligned}$$

subtract: $2C=-8$, $C=-4$

subtract $2D=-4$, $D=-2$

plug in → $A+B=4$, $A-B=2$ ⇒ $A=3, B=1$

So, we have $\dfrac{8x+4}{x^4-1}=\dfrac{3}{x-1}+\dfrac{1}{x+1}+\dfrac{4x+2}{x^2+1}$

⑤ Integrate

$$\to \int \frac{8x+4}{x^4-1}\,dx=\int\frac{3}{x-1}\,dx+\int\frac{1}{x+1}\,dx+\int\frac{4x+2}{x^2+1}\,dx$$

(this can be solved using the fraction building blocks)

__Repeated Factors__

degree 6

degree 7

$$\frac{3x^2+x^6}{(x+9)^3(x^2+x+1)^2}=\frac{A}{x+9}+\frac{B}{(x+9)^2}+\frac{C}{(x+9)^3}+\frac{Dx+E}{x^2+x+1}+\frac{Fx+G}{(x^2+x+1)^2}$$

__Improper Rational Functions__

→ If deg(numerator) ≥ deg(denominator), we have two options:

① Add another polynomial to make up the difference. This polynomial will have degree deg(numerator) − deg(denom)

② Use polynomial long division

→ Example of ①:

$$\frac{x^4-2}{x^2+2x+1}=Ax^2+Bx+C+\frac{D}{x+1}+\frac{E}{(x+1)^2}$$

Multiply each side by $(x+1)^2$ and solve as normal

→ Example of ②

→ we find $(x^4-2)/(x^2+2x+1)=x^2-2x+3-\dfrac{4x+5}{x^2+2x+1}$

→ We solve the integral $\int x^2-2x+3\,dx-\int\dfrac{4x+5}{x^2+2x+1}\,dx$

use PFD on this one

# Improper Integrals

→ There are two types of improper integrals

① Integrating over an infinite domain (ex. $\int_3^\infty e^{-x}\,dx$)

② Integrating over an infinite discontinuity (ex. $\int_{-1}^{2} \frac{1}{x}\,dx$)

## Type 1 Improper Integrals

→ $\int_a^\infty f(x)\,dx$ converges if $\lim_{t\to\infty} \int_a^t f(x)\,dx$ exists

→ Ex $\int_3^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_3^t e^{-x}\,dx = \lim_{t\to\infty} -e^{-x}\Big|_3^t$
$= \lim_{t\to\infty}\left[-e^{-t} + e^{-3}\right] = e^{-3}$

→ So this integral converges to $e^{-3}$

→ There is a similar definition when a limit is $-\infty$

→ Finally $\int_{-\infty}^{\infty} f(x)\,dx$ converges if $\int_{-\infty}^{a} f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ both converge for some $a \in \mathbb{R}$

→ Note that $\int_{-\infty}^{\infty} \sin(x)\,dx \neq \lim_{t\to\infty}\int_{-t}^{t} \sin(x)\,dx = 0$

→ i.e. we don't assume we are tending to $\infty$ and $-\infty$ at the same rate

## P-Test

→ For which values of $p$ does $\int_1^\infty \frac{1}{x^p}\,dx$ converge

→ $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^t = \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}$

→ So, it is $1/(p-1)$ for $p > 1$ and $\infty$ for $p \leq 1$
(since $\int_1^\infty 1/x\,dx = \ln(\infty) - \ln(1) = \infty$)

→ So, we have $\int_1^\infty \frac{1}{x^p}$ converges to $\frac{1}{p-1} \iff p > 1$

## Properties of Type 1 Improper Integrals

-> If $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge, then

① $\int_a^\infty c\, f(x)\,dx = c\int_a^\infty f(x)\,dx$

② $\int_a^\infty f(x) + g(x)\,dx = \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$

③ If $f(x) \le g(x)$ then $\int_a^\infty f(x)\,dx \le \int_a^\infty g(x)\,dx$

④ If $c > a$ then $\int_c^\infty f(x)\,dx$ also converges <u>and</u>

$$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

## Comparison Test

-> It can often be hard to tell if an integral converges, let alone evaluate it. However, the comparison test can help

-> Let f and g be continuous on $[a, \infty)$, and $0 \le f(x) \le g(x)$

-> If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges

-> If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges

-> Ex. does $\int_1^\infty \frac{x^{2/3}-1}{x^3+1}\,dx$ converge or diverge?

-> Good luck finding that antiderivative!

-> However, note that on $[1, \infty)$, $0 \le \frac{x^{2/3}-1}{x^3+1} \le \frac{x^{2/3}}{x^3+1} \le \frac{x^{2/3}}{x^3}$

$\le \frac{1}{x^{7/3}}$. By the p-test, $\int_1^\infty \frac{1}{x^{7/3}}\,dx = \int_1^\infty x^{-7/3}\,dx$ converges, so

$\int_1^\infty \frac{x^{2/3}-1}{x^3+1}\,dx$ must also converge by the comparison test

Remember, we can often just evaluate the function and prove convergence directly

-> Ex. does $\int_3^\infty \frac{1}{x\ln(x)}\,dx$ converge or diverge?

-> On $[3,\infty)$, $\ln(x) > 1 \Rightarrow x\ln(x) > x \Rightarrow \frac{1}{x\ln(x)} \le \frac{1}{x}$

-> So, $\int_3^\infty \frac{1}{x\ln(x)}\,dx \le \int_3^\infty \frac{1}{x}\,dx$.

-> By the p-test, we know that $\int_3^\infty \frac{1}{x}\,dx$ diverges, but this doesn't tell us anything since it is larger than $\int_3^\infty 1/(x\ln(x))\,dx$.

-> In this case, we should actually evaluate:

-> Let $u = \ln(x) \Rightarrow du = 1/x\,dx$

-> $\int_3^\infty \frac{1}{x\ln(x)}\,dx = \int_{\ln(3)}^\infty \frac{1}{u}\,du = \lim_{t\to\infty} \ln|u|\Big|_{\ln(3)}^{t} = \infty$

-> So, $\int_3^\infty \frac{1}{x\ln(x)}$ diverges

## Absolute Convergence

-> Let $f$ be integrable on $[a,\infty)$. If $\int_a^\infty |f(x)|\,dx$ converges, then we say that $\int_a^\infty f(x)\,dx$ converges absolutely.

-> This isn't saying that $\int_a^\infty f(x)$ converges... we need a theorem

-> Absolute convergence theorem (ACT): If $\int_a^\infty |f(x)|\,dx$ converges, then $\int_a^\infty f(x)$ converges

-> Proof: note that $0 \le f(x) + |f(x)| \le 2|f(x)|$.
Since $\int_a^\infty |f(x)|\,dx$ converges, then so does $\int_a^\infty 2|f(x)|\,dx$ and thus, by comparison, $\int_a^\infty f(x) + |f(x)|\,dx$ converges.
Finally, $\int_a^\infty f(x)\,dx = \int_a^\infty f(x) + |f(x)|\,dx - \int_a^\infty |f(x)|\,dx$
Since both of these converge their sum must also converge $\square$.

-> This theorem is mostly used to handle functions that are sometimes negative

-> Ex. does the integral $\int_1^\infty \frac{\sin(x)}{x^3+\sqrt{x}}\,dx$ converge?

-> Note that on $[1,\infty)$, $\left|\frac{\sin(x)}{x^3+\sqrt{x}}\right| \le 1/(x^3+\sqrt{x}) \le \frac{1}{x^3}$
Since $\int_1^\infty 1/x^3\,dx$ converges by the p-test, then $\int_1^\infty \left|\frac{\sin(x)}{x^3+\sqrt{x}}\right|dx$ converges by comparison. Finally, by ACT $\int_1^\infty \frac{\sin(x)}{x^3+\sqrt{x}}$

-> Note that it is possible for f(x) to converge while $\int_a^\infty |f(x)|\, dx$ diverges. An example is $\int_1^\infty \frac{\cos(x)}{x}\, dx$.
-> These integrals are "conditionally convergent"

## Type II Improper Integral

-> Let f be integrable on [a, b) where x=b is a vertical asymptote. We say $\int_a^b f(x)\,dx$ converges if $\lim_{t\to b^-} \int_a^t f(x)\,dx$
-> There is a corresponding right hand limit for an asymptote at the lower bound
-> If f is integrable over [a, b] except at $c \in [a,b]$ where x=c is a vertical asymptote, then $\int_a^b f(x)\,dx$ converges to $\int_a^c f(x)\,dx + \int_b^c f(x)\,dx$ if both pieces converge.

## Type II Examples

$$\to \int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} 2\sqrt{x}\Big|_t^1 = \lim_{t\to 0^+}[2 - 2\sqrt{t}] = 2 - 0 = 2$$

despite integrating over an asymptote that goes to ∞ we have a finite result.

negative result of a positive function

$$\to \int_1^4 \frac{1}{(x-2)^2}\,dx = -\frac{1}{(x-2)}\Big|_1^4 = -\frac{1}{2} - 1 = -\frac{3}{2},$$ which is WRONG!

-> If an asymptote is involved, we need to use a limit.
-> We check $\lim_{t\to 2^-} \int_1^t 1/(x-2)^2\,dx$ first.
-> $\lim_{t\to 2^-} -1/(x-2)\Big|_1^t = \lim_{t\to 2^-} 1/(2-t) - 1 = \infty$
-> Since $\int_1^2 1/(x-2)^2\,dx$ diverges, so does $\int_1^4 1/(x-2)^2$

## P-test II

-> The integral $\int_0^1 \frac{1}{x^p}$ converges to $\frac{1}{1-p} \iff p < 1$
-> Opposite of the regular p-test

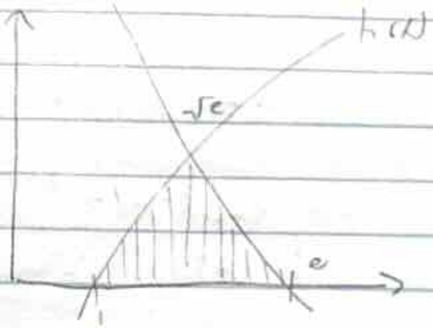
# Applications of Integration

**AREAS** To find the areas between two functions, we create rectangles between f and g, and then take a riemann sum like usual

The area between f and g is $\int_a^b |f(x) - g(x)|\, dx$ over $[a, b]$

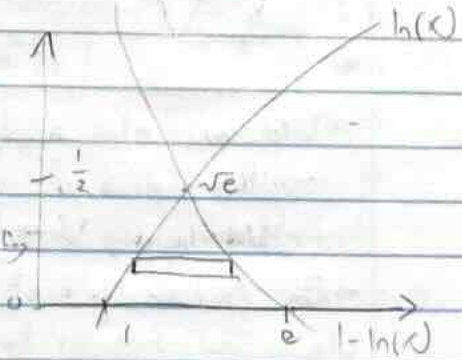
Often it won't be this easy: sometimes we will need to find points of intersection. Sometimes, it is easier to integrate up the y-axis

Ex. Find the area of the triangular region between $y = \ln(x)$, $y = 1 - \ln(x)$ and the x-axis.




-> We need to find where these functions meet, and split into two integrals there. We have $\ln(x) = 1 - \ln(x) \Rightarrow$
$2\ln(x) = 1 \Rightarrow \ln(x) = \frac{1}{2} \Rightarrow x = \sqrt{e}$. So our area is
$$A = \int_1^{\sqrt{e}} \ln(x) - 0\, dx + \int_{\sqrt{e}}^{e} 1 - \ln(x) - 0\, dx$$
$$= [x\ln(x) - x]_1^{\sqrt{e}} + [x - x\ln(x) + x]_{\sqrt{e}}^{e} = \ldots =$$
$$= e - 2\sqrt{e} + 1$$

## Integrating up the y-axis




-> It turns out it is actually easier to integrate along the y-axis in this case

-> The intersection is at 1/2, so we are integrating from 0 to $\frac{1}{2}$.

-> We need to rewrite our equations in terms of y

-> $y = \ln(x) \Leftrightarrow x = e^y$, $y = 1 - \ln(x) \Leftrightarrow x = e^{1-y}$

-> Remember, area in between is [upper graph] - [lower graph]

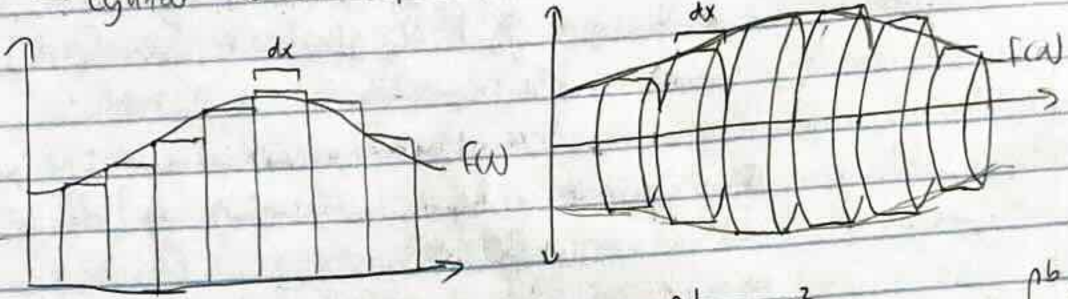
-> So, we have $\int_0^{1/2} e^{1-y} - e^y\, dy = -e^{1-y} - e^y \Big|_0^{1/2} = e - 2\sqrt{e} + 1$

## Volumes of Revolution

-> Recall: the area of a cylinder is $\pi r^2 h$

-> We can wrap a curve around the x-axis, and calculate the volume of this solid by treating each rectangle under the curve as a cylinder.

-> Cylinder has $h = dx$, $r = f(x)$, so its volume is $\pi f^2(x)\,dx$



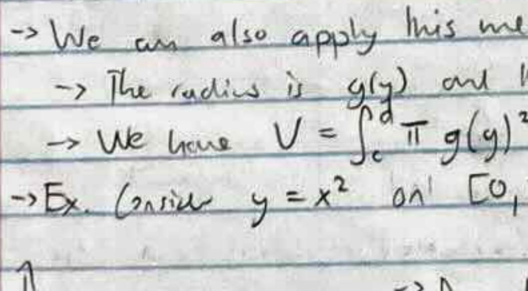

-> This area is described by the integral $\int_a^b [f(x)]^2 \pi \, dx = \pi \int_a^b [f(x)]^2 \, dx$

-> Ex. Calculate the volume of the line $y = x/3$ rotated around the x-axis from $y=0$ to $y=6$

-> $\int_0^6 \pi \left(\frac{x}{3}\right)^2 dx = \frac{\pi}{9}\int_0^6 x^2 \, dx = \frac{\pi}{9} \frac{x^3}{3}\Big|_0^6 = \frac{216\pi}{27} = 8\pi$

-> This is a cone, and it matches the formula for the volume of one

### Washer Method




-> To find the volume of an area between two functions, we subtract the smaller one from the larger one

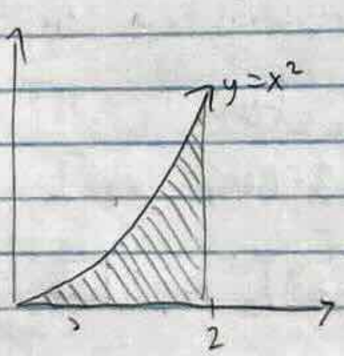
-> $V = \int_a^b \pi f(x)^2 \, dx - \int_a^b \pi g(x)^2 \, dx \neq \int_a^b \pi (f(x) - g(x))^2$

-> We can also apply this method to integrating around the y-axis

-> The radius is $g(y)$ and the height is $dy$

-> We have $V = \int_c^d \pi g(y)^2 \, dy$

-> Ex. Consider $y = x^2$ on $[0, 2]$

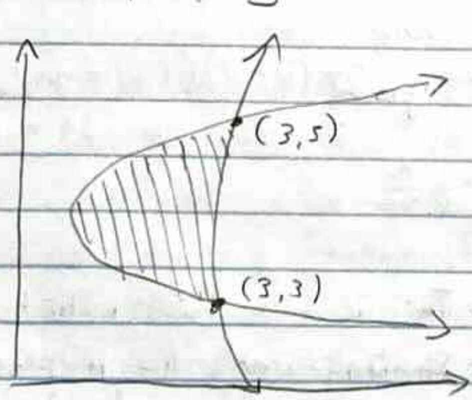



-> Around x-axis: $\int_0^2 \pi (x^2)^2 \, dx = \pi \int_0^2 x^4 \, dx$
$= \pi \left[\frac{1}{5}x^5 \Big|_0^2\right] = \pi\left(\frac{2^5}{5}\right) = 32\pi/5$

-> Around y-axis: $V = \int_0^4 \pi (2)^2 \, dy - \int_0^4 \pi (\sqrt{y})^2 \, dy$
$= 16\pi - \pi \int_0^4 y \, dy = 16\pi - \pi \, 4^2/2 = 16\pi - 8\pi$
$= 8\pi$

## Shell Method

-> Sometimes, using washers is difficult
  -> Horizontal rectangles may be easier than vertical rectangles
  -> This generates a bunch of concentric shells with height $f(x)$
-> We need to find the volume of each shell
  -> It is essentially the outside surface of a cylinder, with a thickness $dy$.
  -> Assume we revolve around the $x$-axis
  -> This surface area is $2\pi y\,(f(y)-g(y))$.
  -> The cylinder has thickness $dy$
-> As such, our integral is $V = \int_c^d 2\pi y\,[f(y)-g(y)]\,dy$
  -> Replace $y$ with $x$ when revolving around the $y$-axis




-> Rotate the region between $x=(y-4)^2+2$ and $x=2(y-4)^2+1$ around the $x$-axis
-> The width of a rectangle is $(y-4)^2+2-(2(y-4)^2+1) = 1-(y-4)^2$
-> So, volume: $2\pi y\,[1-(y-4)^2]\,dy$

-> We have $\int_3^5 2\pi y\left[1-(y-4)^2\right]dy = 2\pi\int_3^5 y-y^3+8y^2-16y\,dy$

$= 2\pi\left(\frac{1}{2}y^2\Big|_3^5 - \frac{1}{4}y^4\Big|_3^5 + \frac{8}{3}y^3\Big|_3^5 - 8y^2\Big|_3^5\right) = \dots = \frac{32\pi}{3}$

## General Rule for Shells

-> (Ignoring rotation axis). Figure out whether you should use horizontal or vertical rectangles.
-> If this choice is parallel to the axis of rotation, use shells, otherwise, use disks or washers

## Rotation around an arbitrary axis

this is like adjusting the location of the axis

-> Instead of rotating around the x or y axis, we may wish to rotate around $x = d$ or $y = k$.

-> For the washer method, if we rotate around $y = k$, we replace $f(x)$ with $k - f(x)$ in all places in the integral

$$\rightarrow \int_a^b \pi \left(g(x)^2 - f(x)^2\right) dx = \pi \int_a^b f(x)^2 dx - \pi \int_a^b g(x)^2 dx \rightarrow \pi \int_a^b (k - f(x))^2 dx + \dots$$

-> For the shell method, it is more complex

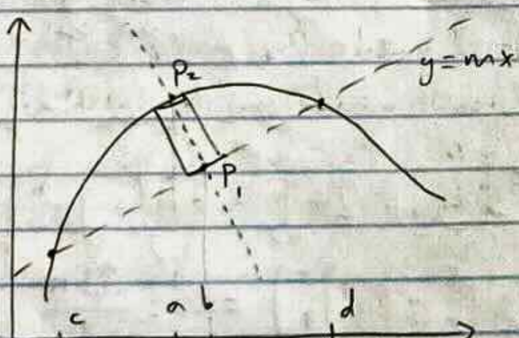
$$\rightarrow \text{Recall: } \int_a^b \overbrace{2\pi x}^{\text{circum}} \overbrace{[f(x) - g(x)]}^{\text{height}} \overbrace{dx}^{\text{thickness}}$$

-> So, when we change the axis, only the circumference (not height) is affected

-> If we rotate around $x = d$, $2\pi x$ will be replaced by $2\pi(x - d)$

$$\rightarrow \text{So, } \int_a^b 2\pi x [f(x) - g(x)]\, dx \rightarrow \int_a^b 2\pi (x - d) [f(x) - g(x)]\, dx$$

## Rotation around a slanted axis




-> This is really hard without more methods

-> Strategy: create line perpendicular to $y = mx$ $\left(y = -\frac{1}{m}x\right)$

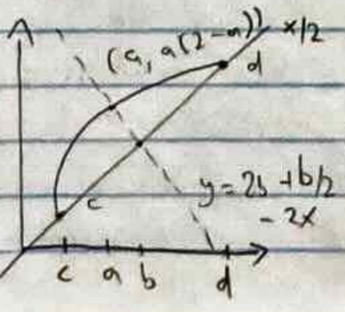
-> Find the points $P_1$ and $P_2$

-> Calculate $h(x) = |P_1 - P_2|$

-> Base of each rectangle is $db = \sqrt{(dx)^2 + (m\,dx)^2} = \sqrt{1 + m^2}\, dx$

$$\rightarrow V = \int_c^d \pi r^2\, db = \pi \sqrt{1 + m^2} \int_c^d h(x)^2\, dx.$$

-> Ex. [hard]: Find the volume when the region $y = x/2$ and $y = x(2 - x)$ is rotated about $y = x/2$




-> Find a in terms of b

-> Find $h(b) = \sqrt{(b - a)^2 + (b/2 - a(2 - a))^2}$

-> $V = \pi \sqrt{1 + m^2} \int_c^d h(x)^2\, dx$ -> $m = 1/2$, c, d are intersections of $x/2$ and $x(2 - x)$

-> Answer: $\dfrac{81\pi\sqrt{5}}{800}$ (somehow)

## Differential Equations

-> Relates an unknown function to its own derivative

-> Ex. $Y(x) = 2Y'(x) - y''(x)$

-> We say the equation $F(x, y, y', y'', \dots, y^{(n)}) = 0$ is called an $n^{th}$-order differential equation

-> Ex. $\sin(y') + 2xy = 3y''$ is a second-order, non-linear DE

-> A DE is considered linear if it can be written in the form $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = f(x)$

-> In this course, we will focus mostly on first-order DEs (both linear and non-linear)

## Solving a simple DE

-> A basic example is $y' = 2y$

-> A function such that its derivative is 2 times itself

-> Recall that $\frac{d}{dx} e^{ax} = ae^{ax}$, so $y = e^{2x}$ is a solution to our DE since $\frac{d}{dx} e^{2x} = 2e^{2x}$

-> In fact, any function of the form $y = Ae^{2x}$ will satisfy the DE. We have $y' = 2Ae^{2x}$ and $2y = 2Ae^{2x}$

-> We call the collection of solutions a <u>family of solutions</u>

-> $y = e^{2x}$, $y = -2e^{2x}$, $y = 0$, $y = -2^{2x+1}$, etc are all members of this family

## Initial Conditions

-> In order to retrieve a specific function in the family of solutions, we need an <u>initial condition</u>

-> We need the value of the function at a certain point

-> Ex. With initial condition $y(0) = 5$, so we set $5 = Ae^{2(0)} = Ae^0 = A \Rightarrow A = 5$.

-> We have found a particular solution

-> A DE coupled with an IC is called an initial value problem (IVP)

## Direction Fields

-> Solving DEs in general often impossible. Often, we can only guess solutions using graphical / numerical techniques

-> A direction field is a collection of line elements on the xy-plane whose slopes correspond with those of the solutions through that point

-> The DE $y' = 2y - x^2$ says that at any point $(a, b)$, the slope of a curve $y(x)$ must be $2b - a^2$.

-> Solutions $y(x)$ will touch each line element tangentially

## Isoclines

-> Sometimes it's better to find isoclines: curves that are not solutions to the DE, but upon which all of the line elements have the same slope.

-> Similar to an isoline in physics.

-> For example, in the DE $y' = 2y - x^2$ setting $y' = 0$ gives $0 = 2y - x^2 \Rightarrow y = \frac{x^2}{2}$ (all line elements around this solution have slope 0

-> In general, for this equation, $y'(x) = m \Rightarrow y = \frac{x^2}{2} + \frac{m}{2}$ isocline

-> As it turns out, the solution to $y' = 2y - x^2$ is $y = \frac{1}{2}x^2 + \frac{x}{2} + \frac{1}{4} + Ce^{2x}$. We can check by taking the derivative.

## Separable DEs

-> Can be written in the form $y' = f(x)g(y) \Rightarrow y' \frac{1}{g(y)} = f(x)$

-> Ex. $y' + y^3 = \ln(y^2+1)$, $y' = e^{x-y}$ (since $e^{x-y} = e^x e^{-y}$)

-> $y' + \sin(x)y = 3y^2$, $y' = 2y - x$ is not separable

-> Note that if the constant function $y(x) = b$ satisfies $g(b) = 0$, then $y(x) = b$ is a solution to $y' = f(x)g(y)$

-> We call solutions $y(x) = b$ to $y' = f(x)g(y)$ the equilibrium solutions since b is unchanging.

-> Ex. $y' = (y^2 - 4)(x+3)$ has equilibrium solutions $y(x) = 2, -2$ since 2 and $-2$ are the roots of $y^2 - 4 = 0$.

## Solving Separable DEs

1. Find all equilibrium solutions
2. Rewrite as $h(y)\frac{dy}{dx} = f(x)$ (where $h(y) = \frac{1}{g(y)}$)
3. Integrate both sides with respect to $x$

$$\int h(y)\frac{dy}{dx}\,dx = \int f(x)\,dx \quad \rightarrow \int h(y)\,dy$$

this follows from the chain rule, and is the same thing we did for u-sub

4. If possible, isolate for $y(x)$. Otherwise, solutions will remain in implicit form

## Separable DE Example

-> Solve the separable DE $\frac{dy}{dx} = 2x(y-1)$

-> Note that $y=1$ solves the DE (equilibrium solution)

-> Assume that $y(x) \neq 1$.

-> We get $\int \frac{1}{y-1}\,dy = \int 2x\,dx \Rightarrow \ln|y-1| = x^2 + C$

$$\Rightarrow |y-1| = e^{(x^2+C)} = e^C \cdot e^{x^2} = C \cdot e^{x^2}$$

-> Next, we move the absolute value by using $\pm$

$$\Rightarrow y-1 = \pm[C \cdot e^{x^2}] = \pm Ce^{x^2}. \text{ Let } A = \pm C.$$

$\Rightarrow y = 1 + Ae^{x^2}$. Notice that $A \neq 0$ since $e^C \neq 0$ for all $C$.

-> So, our two solutions are $y(x) = 1$ and $y(x) = 1 + Ae^{x^2}$, $A \neq 0$.

-> If we (sacrilegiously) let $A = 0$, we get $y(x) = 1$!

-> So, we can combine these into $y(x) = 1 + Be^{x^2}$ for $B \in \mathbb{R}$

Exercise / Example 2 - Separable DEs

-> Solve $(x^2+1)y' = xy$ , $y(0)=2$

-> $\frac{1}{y}y'$

## Applications of DEs

Mixing Tank

-> Your shark can live in water with a chlorine concentration of up to 5mol/L

-> Your 15L tank has 10L of solution with concentration 0.3mg/L

-> You turn on the inflow pump and set the speed to 4L/min with a concentration of 6mol/L

-> Drain is also set to 4L/min to avoid overflow

-> You wait 2 minutes

-> Is your shark alive?!?

Solution

-> We have $\frac{dm}{dt}$ = mol rate in − mol rate out

-> Equation is in unit mol/min (we will do dimensional anal.)

-> According to the question, mass rate in = vol rate in * mol per vol $= 4L/min \cdot 6\,mol/L = 24\,mol/min$

-> Mol rate out = vol rate out · mass per vol $= 4L/min \cdot \frac{m(t)}{10}$ unknown mols

$= 2m(t)/5$ mol/min

-> So, $dm/dt = 24 - 2m/5$. This is a separable DE

-> We also know the initial concentration is 0.3 mg/L in 10L of solution, so $m(0) = 10(0.3) = 3$ mg

-> So, we wish to solve $\frac{dm}{dt} = 24 - \frac{2m}{5}$, $m(0) = 3$ and compute $m(2)$.

-> We have $\frac{dm}{dt} = 24 - \frac{2m}{5} \Rightarrow \frac{dm}{dt} = \frac{120 - 2m}{5}$

$$\Rightarrow 5\, dm = 120 - 2m\, dt \Rightarrow \frac{1}{120-2m} dm = \frac{1}{5} dt$$

$$\Rightarrow -\frac{1}{2} \ln |120 - 2m| = \frac{t}{5} + C. \Rightarrow -\frac{2t}{5} + C$$

$$\Rightarrow 120 - 2m = Ae^{-2t/5},\ A = \pm e^c,\ \Rightarrow m(t) = 60 + Be^{-2t/5}$$

-> Using $m(0) = 3$ we get $3 = 60 + B \Rightarrow B = -57$, so $m(t) = 60 - 57e^{-2t/5}$

Linear DEs

-> General, first order linear DE: $a(x)y' + b(x)y = c(x)$
  -> standard form: $y' + P(x)y = Q(x)$

-> If $P(x)$ and $Q(x)$ are continuous on $I$, then for each $(x_0, y_0)$ where $x_0 \in I$, the IVP $y' + P(x)y = Q(x)$, $y(x_0) = y^0$ has exactly one solution

-> To do this, we need an integrating factor
  -> Converts the desired # of expressions into something that can be directly integrated
  -> We want to convert the LHS into something like $d/dx\,[F(x, y(x))]$ so integrating removes the $d/dx$
  -> Multiply by $I(x)$ to get $I(x)y' + I(x)P(x)y = I(x)Q(x)$
    -> We define $I(x)$ such $d/dx\,[I(y)] = I(x)y' + I(x)p(x)y$ (using the product rule)

-> So, we have $\frac{d}{dx}[I(x)y] = I(x)Q(x)$ ← only depends on $x$

-> So, $I(x)y = \int I(x)Q(x)\,dx + C \Rightarrow y = \frac{1}{I(x)}\int I(x)Q(x)\,dx + \frac{C}{I(x)}$
  -> $I(x)$ always exists (woah!)

Finding $I(x)$

-> Remember $\frac{d}{dx} I(x) y = I(x) y' + I(x) p(x) y$

$\Rightarrow I'(x) y = I(x) p(x) y \Rightarrow I'(x) = I(x) p(x)$

-> This is a DE for $I(x)$!

-> $\frac{dI}{dx} = I(x) p(x) \Rightarrow \int \frac{1}{I(x)} dI = \int p(x) dx \Rightarrow \ln|I(x)| = \int p(x) dx + C$

-> $I(x) = Ae^{\int p(x) dx}$, $A = I(x) e^{C} \neq 0$.

-> Any function in the form $I(x) = Ae^{\int p(x) dx}$ will force the DE into the form $d/dx(I(x)y) = I(x) Q(x)$

-> So: $I(x) = e^{\int p(x) dx}$

Example of Linear DE -> Solve $y' = x - 2y$

-> In standard form: $y' + 2y = x$

-> So $I(x) = e^{\int p(x) dx} = e^{\int 2 dx} = e^{2x}$ (plus C not needed)

-> Multiply by $I(x)$: $e^{2x} y' + 2e^{2x} y = xe^{2x}$

$\Rightarrow \frac{d}{dx}(e^{2x} y) = xe^{2x} \Rightarrow e^{2x} y = \int x e^{2x} dx$

$\Rightarrow e^{2x} y = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2} dx \Rightarrow e^{2x} y = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$

-> $y(x) = \frac{x}{2} - \frac{1}{4} + Ce^{-2x}$

$\boxed{\text{Application: Banking}}$

-> An account is compounded at 4% per year. If the account initially holds \$100 and deposits are made at a rate of \$50t per year (continuously), how much money will there be in 10 years.

-> Let $B(t)$ denote the amount at time $t$.

-> Continuous compounding suggests $\frac{dB}{dt} = rB$, where $r = 0.04$ in this case

-> Remember, we are also depositing, so we have

$\frac{dB}{dt} = rB + 50t$ (this is an LDE).

-> Can be solved with $I(t) = e^{-rt}$ yielding $B(t) = Ce^{rt} - \frac{50}{r^2} - \frac{50t}{r}$

-> Initial condition $B(0) = 100 \Rightarrow C = 31350$, so $B(10) = 3018.70$.

## Application: Population Models

- In most population models, the rate of change is proportional to the size of the population
  - Mathematically: $\frac{dP}{dt} = kP$, where $k$ is the rate constant
  - $k > 0 \Rightarrow$ more births than deaths, $k < 0 \Rightarrow$ opposite
  - $dP = kP\,dt \Rightarrow \frac{1}{P}\,dp = k\,dt \Rightarrow \int \frac{1}{P}\,dp = \int k\,dt$
  - $\Rightarrow \ln|P| = kt + C \Rightarrow |P| = e^{kt+C} \Rightarrow P = Ae^{kt}$
  - Letting $P_0 = 0$ yields $P(t) = P_0 e^{kt}$

## Logistic growth

- The previous model is inefficient/inaccurate because populations don't grow exponentially without bounds
  - They have a carrying capacity (usually $M$)
  - Better model: $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ $(k > 0)$
- If $P < M$, $\frac{P}{M} < 1$, so $\frac{dP}{dt} > 0$ ($P$ is growing)
  - Reverse is also true: $P > M$ means $P$ is shrinking
  - As $P$ approaches $M$, the curve starts to flatten

## Solving Logistic growth

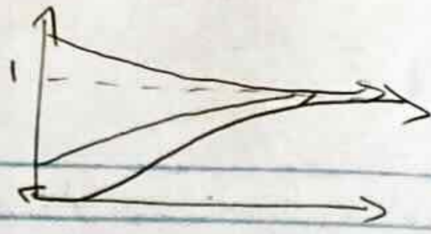
- Notice $P = 0$ and $P = M$ are equilibrium solutions
- $\frac{dP}{dt} = kP\left(\frac{M-P}{M}\right) \Rightarrow \int \frac{M}{P(M-P)}\,dP = \int k\,dt$
- Using PFD, we have $\frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P}$ so

we have $\int \frac{1}{P}\,dP + \int \frac{1}{M-P}\,dP = \int k\,dt \Rightarrow \ln|P| - \ln|M-P| = kt + C$

$$\Rightarrow \ln\left|\frac{P}{M-P}\right| = kt + C \Rightarrow \frac{P}{M-P} = Ae^{kt} \Rightarrow Pe^{-kt} = AM - AP$$

$$\Rightarrow Pe^{-kt} + AP = AM \Rightarrow P = \frac{AM}{A + e^{-kt}}.$$

- Letting $P(0) = P_0 \Rightarrow P_0 = A(M - P_0)$
- So: $P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-kt}}$



→ Notice that $\lim_{t\to\infty} P(t) = M$

→ The population will approach its carrying capacity

→ $P(t) = 0$ is an unstable equilibrium

→ $P(t) = M$ is a stable equilibrium

## Application: Newton's law of cooling

→ Change in temperature is proportional to the difference between the temperature of an object and its surroundings.

→ $\frac{dT}{dt} = k(T_0 - T)$ $k>0$ (separable and linear)

## Application: Fluid Flow

→ A tank of liquid with cross-sectional area is leaking liquid through a hole of cross-sectional area $A_h$, then the rate of change of the height of the liquid is

$\frac{dh}{dt} = -\frac{A_h}{A_c}\sqrt{2gh}$, where $g$ is gravity

## Application: Aerodynamics

→ Speed, $v$ of a falling object with mass $m$ subject to a drag coefficient $\alpha$ can be modelled by

→ $m \cdot \frac{dv}{dt} = mg - \alpha v$

## Series (Intro)

-> Given a sequence $\{a_n\}$, a series is when all the terms of $a_n$ are added (i.e. $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \dots$)

  -> In this class, series $\Leftrightarrow$ infinite series

-> Goal: studying convergence properties of series

  -> Occasionally, we also want to know what it adds to

  -> This second question is often really hard

## Series as Sequences

-> Series can be defined as sequences, specifically, the sequence of partial sums up to $n$

  -> $S_n = \sum_{k=1}^{n} a_k$ ($n^{th}$ partial sum of the series)

  -> Ex. $a_n = \frac{2}{3^n} \Rightarrow S_1 = a_1 = \frac{2}{3}$, $S_2 = a_1 + a_2 = \frac{2}{3} + \frac{2}{9} = \frac{8}{9}$, etc.

-> The sequence of partial sums defines convergence

## Convergence and the Geometric Series

-> The series $\sum_{n=1} a_n$ converges $\Leftrightarrow$ $\{S_n\}$ converges

  -> So, $\lim_{n \to \infty} S_n$ exists

-> Otherwise, the series diverges (kind of like improper integrals)

-> Usually, it is difficult to express the partial sums as an explicit formula (so we can't just take the limit)

-> There is an important exception: the geometric series

  -> Geometric Series: $\sum_{n=0}^{\infty} Ar^n$ (where $A$ is constant)

  -> $S_n = A + Ar + Ar^2 + Ar^3 + \dots + A^n$ ($n^{th}$ partial sum)

  $\Rightarrow rS_n = Ar + Ar^2 + Ar^3 + \dots + A^{n+1}$

  $\Rightarrow S_n - rS_n = A + Ar^{n+1} \Rightarrow S_n = \frac{A(1-r^{n+1})}{1-r}$

-> So, $\sum_{k=0}^{n} Ar^k = \frac{A(1-r^{n+1})}{1-r}$

-> The series will converge as long as $\lim_{n \to \infty} S_n$ exists

-> Notice that if $r > 1$, $r^n \to \infty$ so the geometric series diverges

  -> If $r < -1$, $r^n \to$ DNE (oscillates $|r^n| \to \infty$)

  -> $(-1 < r < 1)$, $r^n \to 0$,

## Limit of the Geometric Series

-> If $|r^n| < 1$ :

-> $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{A(1-r^{n+1})}{1-r} = \frac{A}{1-r}$

-> Often remembered as First term / (1 - common ratio)

## Special Cases: Geometric Series

-> Special cases: $r = -1, 1$

-> $\sum_{n=0}^{\infty} A(-1)^n = A - A + A - A + \dots$, $\{S_n\} = \{A, 0, A, 0 \dots\}$

-> This clearly diverges (we cannot find an $(L-\varepsilon, L+\varepsilon)$)

-> $\sum_{n=0}^{\infty} A = A + A + A + \dots$ $S_n = (n+1)A$

-> This clearly diverges to $\infty$

## Computing Geometric Series

-> The series $\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \dots$ has $A = \frac{2}{3}$ and $r = \frac{1}{3}$, so it adds up to $(\frac{2}{3})/(1-\frac{1}{3}) = (2/3)/(2/3) = 1$

-> Notice we started at $n=1$, but starting at any $n$ would still yield a convergent series

-> It will (almost certainly) add up to something else

-> Ex A bouncy ball is dropped from 3m high and rebounds to 80% of its height after each bounce

-> $d_0 = 3$, $d_1 = 2.4(2) = 3(0.8)2$, $d_2 = 3(0.8)^2 2$

-> How far does the ball travel? Take the sum!

different first term

-> $d = \sum_{i=0}^{\infty} d_i = d_0 + \sum_{i=1}^{\infty} d_i = 3 + \sum_{i=1}^{\infty} 3(\frac{4}{5})^i(2) = 3 + 6\sum_{i=1}^{\infty}(\frac{4}{5})^i$

$= 3 + 6\left[\frac{4/5}{1-4/5}\right] = 6\left[\frac{4/5}{1/5}\right] + 3 = 6(4) + 3 = 27$

## Divergence Test

-> If $\sum_{n=k}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$

-> The contrapositive is more helpful: the divergence test

-> If $\lim_{n\to\infty} a_n \neq 0$ (or DNE), then $\sum_{n=k}^{\infty} a_n$ diverges (for any $k$)

## Proof of Divergence Test

-> We wish to prove that $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

-> Let $S_n = \sum_{k=1}^{n} a_k$

-> Note that $S_n - S_{n-1} = a_n + a_{n-1} + \dots + a_1 - (a_{n-1} + a_{n-2} + \dots + a_1)$

-> This is all equal to $a_n$

-> Since $\sum a_n$ converges, then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = L$.

-> So, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} [S_n - S_{n-1}] = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0$

-> We have proven our statement

## Harmonic Series

-> Is the converse true?

-> I.e. $\lim_{n\to\infty} a_n = 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ converges (true?)

-> Nope!

-> Consider the harmonic series : $a_n = \frac{1}{n}$

-> We have $\lim_{n\to\infty} \frac{1}{n} = 0$ (obviously)

-> However, note the following about the harmonic series

-> $S_1 = 1$

-> $S_2 = 1 + \frac{1}{2}$

-> $S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2} = 2$

-> $S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$
$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{5}{2}$ [etc]

-> So, in general, $S_{2^n} > 1 + \frac{n}{2}$

-> This means, for any $k$, we can find some $n$ $(k = 1 + \frac{n}{2})$ such that $S_n > k$

-> In other words, $\lim_{n\to\infty} 1 + \frac{n}{2} = \infty$ and we can always find some $n$ such that $S_n > 1 + \frac{n}{2}$, so $\lim_{n\to\infty} S_n = \infty$ as well

-> This disproves the converse of the divergence test

Starting at a random Index

-> Convergence only depends on end behavior

-> Where you start is completely irrelevant

-> Assume $a_n$ is defined for all $n$

-> If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=j}^{\infty} a_n$ converges for all $j$

-> If $\sum_{n=j}^{\infty} a_n$ converges for some $j$ then $\sum_{n=1}^{\infty} a_n$ converges

-> I.e. Finitely many terms does not affect convergence.

Series Arithmetic

-> If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then

① $\sum_{n=1}^{\infty} c a_n$ converges to $c\sum_{n=1}^{\infty} a_n$ for any $c \in \mathbb{R}$

② $\sum_{n=1}^{\infty} [a_n + b_n]$ converges to $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

-> Note: if $\sum a_n$ or $\sum b_n$ diverges $\sum[a_n + b_n]$ diverges

-> Note: if both $\sum a_n$ and $\sum b_n$ diverge, we don't know whether $\sum [a_n + b_n]$ diverges

-> Divergence doesn't necessarily mean $\to \pm\infty$

-> One could cancel another out

-> This is similar to integration (specifically improper integration)

-> Both are adding up infinitely many terms, integration just has a finer grain

-> Ex. $\sum \frac{1}{n}$ and $\sum \frac{2}{n}$ both diverge

-> $\sum \frac{1}{n} + \sum \frac{2}{n} = \sum \frac{3}{n}$, which clearly diverges

-> Ex. $\sum \frac{1}{n}$ and $\sum -\frac{1}{n}$ both diverge

-> $\sum \frac{1}{n} + \sum -\frac{1}{n} = \sum \frac{1}{n} - \frac{1}{n} = \sum 0 = 0$, which converges

-> Ex. $\sum \sqrt[n]{e}$ and $\sum -\sqrt[n+1]{e}$ both diverge (by the divergence test)

-> However, $\sum [\sqrt[n]{e} - \sqrt[n+1]{e}]$ converges! How?

## Telescoping Series

-> When a piece of $a_k$ cancels with $a_{k+1}$, we call this a telescoping series

-> The series collapses in on itself like a telescope

Ex

-> $\sum_{k=1}^{n} \left[e^{1/k} - e^{1/k+1}\right] = e^1 - e^{\frac{1}{2}} + (e^{1/2} - e^{1/3}) + (e^{1/3} + e^{1/4}) + \ldots$

-> Each $e^{1/n}$ will cancel another

-> So, the $n$th partial sum is $e^1 - e^{1/n+1} = e + \sqrt[n+1]{e}$

-> This extends to the infinite series as well!

-> Since $\lim_{n\to\infty} S_n = e - 1$, then $\sum_{n=1}^{\infty} e^{1/n} + e^{1/(n+1)}$ converges

-> Specifically, it converges to $e-1$

Ex

-> $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$. The divergence test $\lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) = 0$ doesn't help

-> However, we can still show it diverges

-> $S_n = \sum_{k=1}^{n} [\ln(k) - \ln(k+1)] = \ln(1) + \ln(2) + (\ln(2) - \ln(3)) + \ldots$

-> This telescopes to $\ln(1) - \ln(n+1) = -\ln[n+1]$

-> $\lim_{n\to\infty} -\ln(n+1) = -\infty$, so $\sum \ln\left(\frac{n}{n+1}\right)$ diverges

## Integral Test

-> Not all series are geometric or telescoping, and the divergence test only tests for ... divergence

-> We need something stronger : Integral test

-> Given $\sum_{n=1}^{\infty} a_n$, if we can find a positive, eventually decreasing continuous function $f(x)$ where $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

Ex. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because $\int_1^{\infty} \frac{1}{x}\,dx$ diverges

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges

-> Note that these do not converge to the same value

## P-series

-> We can port the p-test to infinite sums

-> $\sum_{n=1}^{\infty} \frac{1}{n^p}$ will converge if and only if $p > 1$

## Error Estimates

-> Let $f(x)$ satisfy the integral test and $\sum_{n=1}^{\infty} a_n = L$

-> Let the $n^{th}$ partial sum $S_n$ be $S_n = \sum_{i=1}^{n} a_i$

-> We wish to find the error by approximating $\sum_{i=1}^{\infty} f(i)$ with $S_n$

-> We have $L = \sum_{i=1}^{\infty} a_i = \sum_{i=1}^{n} a_i + \sum_{i=n+1}^{\infty} a_i = S_n + \sum_{i=n+1}^{\infty} a_i$

-> The error/remainder $R_n$ is $R_n = L - S_n = \sum_{i=n+1}^{\infty} a_i$

-> We have $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$

-> Or $S_n + \int_{n+1}^{\infty} f(x)\,dx \leq L \leq S_n + \int_{n}^{\infty} f(x)\,dx$

-> Ex. Find an interval for the value of $L = \sum_{n=1}^{\infty} \frac{1}{n^2}$ if we approximate it with 10 terms

-> $S_n + \int_{n+1}^{\infty} \frac{1}{x^2}\,dx \leq L \leq S_n + \int_{n}^{\infty} \frac{1}{x^2}\,dx$

$\Rightarrow S_n + \left(-\frac{1}{x}\right)\Big|_{n+1}^{\infty} \leq L \leq S_n + \left(-\frac{1}{x}\right)\Big|_{n}^{\infty}$

$\Rightarrow S_n + \frac{1}{n+1} \leq L \leq S_n + \frac{1}{n} \Rightarrow S_{10} + \frac{1}{11} \leq L \leq S_{10} + \frac{1}{10}$

-> Ex. What is the upper bound in error in using $S_5$ to approximate $\sum_{n=1}^{\infty} \frac{1}{n^4}$

-> $R_5 \leq \int_{5}^{\infty} \frac{1}{x^4} = -\frac{1}{3x^3}\Big|_{5}^{\infty} = \frac{1}{3(5)^3} = \frac{1}{375}$

-> What is the smallest $n$ such that, the upper bound on the error in using $S_n$ to approximate $\sum_{k=1}^{\infty} k^{-3}$ is less than 0.001

-> Answer: $n = 23$

## Proof of Integral Test

-> Let $S_M = a_1 + a_2 + \dots + a_M$, $S_N = S_n + a_{n+1} + \dots + a_M$

-> We have $S_n + \int_{n+1}^{M} f(x)\,dx \le S_M \le S_n + \int_{n}^{M} f(x)\,dx$

-> If $\int_1^\infty f(x)\,dx$ converges, we have $\int_1^\infty f(x)\,dx = \alpha$

=> $S_M \le S_n + \alpha$

-> Since each $a_n \ge 0$, $\{S_M\}$ must be an increasing sequence, so $\lim_{M \to \infty} S_M$ exists by the MCT.

-> So, $\sum_{n=1}^{\infty} a_n$ converges

-> Divergence: $\int_1^\infty f(x)\,dx$ diverges => $\lim_{M \to \infty} \int_{n+1}^{M} f(x)\,dx = \infty$ since $f(x) \ge 0$. Since $S_M \ge \int_{n+1}^{M} f(x)\,dx$, then $\lim_{M \to \infty} S_M = \infty$, so the series diverges

## Comparison Test

-> Consider the series $\sum_{n=1} \frac{1}{n^2+1}$ (this is not a p-series)

-> However, note that $1/n^2+1 < 1/n^2$ for all $n$ and $\sum_{n=1}^{\infty} 1/n^2$ converges

-> Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ must converge as well

-> Let $0 \le a_n \le b_n$ for all $n$ after some point $N$

-> $\sum_{n=1}^{\infty} b_n$ converges => $\sum_{n=1}^{\infty} a_n$ converges

-> $\sum_{n=1}^{\infty} a_n$ diverges => $\sum_{n=1}^{\infty} a_n$ diverges

## Comparison test examples

-> does $\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-1}$ converge or diverge?

-> $\frac{n^2+1}{n^3-1} > \frac{n^2+1}{n^3} > \frac{n^2}{n^3} = \frac{1}{n}$. $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so $\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-1}$ does too

-> $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ : $\frac{1}{\ln(n)} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ does too

## Convergence Test Examples continued

→ $\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2}$ ? We have $\frac{1+\sin(n)}{n^2} \le \frac{2}{n^2}$ Since $\sum \frac{1}{n^2}$ converges, $2\sum \frac{1}{n^2} = \sum \frac{2}{n^2}$ converges, so by the comparison test, $\sum \frac{1+\sin(n)}{n^2}$ converges

→ $\sum (5^n + n)/(4-\frac{1}{n})^n$ ? We have $(5^n+n)/(4-\frac{1}{n})^n > \frac{5^n}{(4-\frac{1}{n})^n} > \frac{5^n}{4^n}$. Since $\left(\frac{5}{4}\right)^n$ diverges (since it is a geometric w/ $r>1$) $\sum \frac{5^n+n}{(4-\frac{1}{n})^n}$ diverges

## Limit Comparison Test

solve using process of elimination

→ Lets consider $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$. Since $n^2-1 < n^2$, we get $1/(n^2-1) > (1/n^2)$. Since $\sum 1/n^2$ converges, this inequality is not helpful

→ but it still acts like $\frac{1}{n^2}$, right?

→ This problem is solved by the limit comparison test

→ Given $a_n \ge 0$ and $b_n \ge 0$

- → If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $0 < L < \infty$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge
- → If $\lim_{n\to\infty} a_n/b_n = 0$ AND $\sum b_n$ converges then $\sum a_n$ converges
- → If $\lim_{n\to\infty} a_n/b_n = \infty$ AND $\sum b_n$ diverges then $\sum a_n$ diverges

→ Usually, $a_n$ is what we want to determine and we pick a series $b_n$ that we know something about

## Limit Comparison Test examples

In general, we ignore things that are small as $n \to \infty$

$\to \sum_{n=2}^{\infty} \frac{1}{n^2-1}$. Let $a_n = \frac{1}{n^2-1}$ and $b_n = \frac{1}{n^2}$. So, $\lim_{n\to\infty} \frac{a_n}{b_n}$

$= \lim_{n\to\infty} \frac{n^2}{n^2-1} = \frac{1}{1+0} = 1$. By the LCT, since $\lim_{n\to\infty} \frac{a_n}{b_n}$

converges, so does $\sum \frac{1}{n^2-1}$

$\to \sum_{n=1}^{\infty} \frac{2n^2+1}{\sqrt{1+n+n^6}}$. This behaves like $\frac{2n^2}{\sqrt{n^6}} \sim \frac{1}{n}$ for large $n$

Choose $b_n = \frac{1}{n}$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(2n^2+1)n}{\sqrt{1+n+n^6}} = \lim_{n\to\infty} \frac{2n^3+n}{\sqrt{1+n+n^6}}$

$= \lim_{n\to\infty} \frac{(n^3)(2+1/n^2)}{(n^3)\sqrt{1/n^6+1/n^5+1}} = \lim_{n\to\infty} \frac{2+1/n^2}{\sqrt{1/n^6+1/n^5+1}} = 2$. By LCT,

since $\sum b_n = \sum \frac{1}{n}$ diverges, so too does $\sum_{n=1}^{\infty} \frac{2n^2+1}{\sqrt{1+n+n^6}}$

$\to \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$. $a_n = \frac{\ln(n)}{n^2}$. $b_n = \frac{1}{n^2}$ doesn't work because $1/n^2$

doesn't diverge and (LCT doesn't apply). $b_n = \frac{1}{n}$

yields: $\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \frac{\ln(n)}{n} = 0$, which still doesn't

help since $b_n = \frac{1}{n}$ diverges.

## Alternating Series

$\to$ Series of the form $\sum_{n=1}^{\infty} (-1)^n a_n = a_1 - a_2 + a_3 - a_4 \dots$ if $a_n \geq 0$ for all $n$

$\to$ These usually behave better than their regular counterparts

## Alternating Series Test

$\to$ Given $\sum(-1)^n a_n$ if

1. $a_n \geq 0$ for all $n \geq$ some $N$
2. $a_{n+1} \leq a_n$ for all $n \geq$ some $N$
3. $\lim_{n\to\infty} a_n = 0$

$\to$ Then $\sum(-1)^n a_n$ converges

# Alternating Series examples

-> Alternating harmonic series : $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$

-> This converges since $a_n = 1/n$ satisfies the AST requirements

-> In fact, this converges to $\ln(2)$

-> Does $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\ln(n+1)}$ converge?

-> $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\ln(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln(n+1)}$

-> $1/\ln(n+1)$ satisfies the requirements, so it does converge

-> Does $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}}$ converge?

-> We have $\lim_{n \to \infty} \frac{\ln(n)}{\sqrt{n}} = 0$ and $a_n > 0$ for all $n \geq 1$

-> Is $a_n$ decreasing?

-> $\frac{d}{dx}\left[\frac{\ln(x)}{\sqrt{x}}\right] = \frac{\frac{1}{x}\sqrt{x} - \frac{\ln(x)}{2\sqrt{x}}}{x} = \frac{2 - \ln(x)}{2x\sqrt{x}}$

$\frac{2 - \ln(x)}{2x\sqrt{x}} > 0 \Rightarrow 2 - \ln(x) > 0 \Rightarrow -\ln(x) > -2$

$\Rightarrow \ln(x) > 2 \Rightarrow x > e^2$

-> So $a_{n+1} > a_n$ for all $n \geq 8$

-> Does $a_n = e^{1/n}$ converge — if $a_n$ is an alternating series

-> We have $\lim_{x \to \infty} e^{1/x} = e^{1/\infty} = e^0 = 1$

-> This is not 0, so the AST can't be used

-> By the divergence test, $\sum_{n=1}^{\infty} (-1)^n e^{1/n}$ diverges

## Proof of AST

-> Assume we have $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ with $a_n \geq 0$, $a_{n+1} < a_n$ and $\lim_{n \to \infty} a_n = 0$.

-> In general, $S_{2n} \geq S_{2n-2} \geq \ldots \geq S_4 \geq S_2$

-> Therefore $\{S_{2n}\}$ is increasing

-> However, $S_{2n} < a_1$, which means it is bounded above

-> By MCT, it must converge

-> Let $\lim_{n \to \infty} S_{2n} = S$

-> We have $S_{2n+1} = S_{2n} + a_{n+1} \Rightarrow \lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} S_{2n} + \lim_{n \to \infty} a_{n+1}$
$= S + 0 = S$

-> Since both even and odd partial sums converge to the same value we have $\lim_{n \to \infty} S_n = S$

-> Therefore, $\sum (-1)^{n+1} a_n$ converges.

## AST Error Estimate

-> We have

-> $|R_n| = |S - S_n| < a_{n+1}$

-> If $a_{n+1} > 0$ then $S_n$ is an underestimate

-> If $a_{n+1} < 0$ then $S_n$ is an overestimate

-> Therefore, the error of the partial sum up to $n$ is no more than the magnitude of the first omitted term

-> Ex. Error of $S_9$ approximating $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$

-> $S_9 = 0.7456$

-> error $= |S - S_9| < a_{10} = \frac{1}{10} \Rightarrow S = 0.7456 \pm 0.1$

-> since $n = 10$ and $(-1)^{10+1} = (-1)^{11} = -1 < 0$, $S_9$ is an overestimate

-> So, $S \in (0.6456, 0.7456)$

## Absolute and Conditional Convergence    Given $\sum_{n=1}^{\infty} a_n$

① $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent

② $\sum_{n=1}^{\infty} |a_n|$ diverges but $\sum_{n=1}^{\infty} a_n$ converges
$\Rightarrow a_n$ is conditionally convergent

→ If $\sum a_n$ absolutely, then it also converges
→ If $\sum |a_n|$ converges then so does $\sum a_n$
→ Proof:
→ Note that $-|a_n| \le a_n \le |a_n| \Rightarrow 0 \le a_n + |a_n| \le 2|a_n|$
→ Since $\sum |a_n|$ converges then $\sum 2|a_n|$ converges,
so $\sum a_n + |a_n|$ converges by the comparison test
→ We have $\sum a_n = \sum a_n + |a_n| - \sum |a_n|$
→ Both of the terms converge so $\sum a_n$ must also converge
→ Examples
→ $\sum (-1)^n \left(\frac{1}{n}\right)$ converges conditionally / $\sum 1/n$ diverges, $\sum (-1)^n/n$ converges
→ $\sum (-1)^n \left(\frac{1}{n^2}\right)$ converges absolutely — $\sum 1/n^2$ converges

## Ratio Test    Given $\sum_{n=1}^{\infty} a_n$, let $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

If you come across factorials, the ratio test should be used since ? is n+1-able

→ $L < 1 \Rightarrow \sum a_n$ is absolutely convergent
→ $L > 1 \Rightarrow \sum a_n$ diverges
→ $L = 1 \Rightarrow$ we don't know anything (inconclusive)
→ another test will be necessary
→ Remember, the limit comparison test doesn't have any problems with the value of 1 (just the ratio test)
→ Examples
→ $\sum 2^n/n! \to \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}/(n+1)!}{2^n/n!}\right| = \left|2\frac{1}{(n+1)}\frac{2^n/n!}{2^n/n!}\right|$
$= \left|2\left(\frac{1}{n+1}\right)\right|$, $\lim_{n\to\infty} |2/n+1| = 0$, so $\sum 2^n/n!$ is absolutely convergent
→ In fact $\sum b^n/n!$ converges for all $b$
→ $\sum (-1)^n/n^2 \to \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}/(n+1)^2}{(-1)^n/n^2}\right| = \left|\frac{n^2}{(n+1)^2}\right| = \left|\frac{n^2}{n^2+2n+1}\right|$
$\lim_{n\to\infty} \left|\frac{n^2}{n^2+2n+1}\right| = 1$, so the ratio test is inconclusive

-> The ratio test always fails against a ratio of polynomials
-> Even when fractional exponents are involved
-> However, the LCT often works in these scenarios

-> Ex. Does $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converge?

-> ratio : $\left|\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n}\right| = (n+1)\left|\frac{n!/(n+1)^n(n+1)}{n!/n^n}\right|$

$= \left|\frac{n!/(n+1)^n}{n!/n^n}\right| = \left|\frac{n!}{n!}\cdot\frac{n^n}{(n+1)^n}\right| = \left|\frac{n^n}{(n+1)^n}\right| = \left|\frac{n}{n+1}\right|^n$

$= \left|\frac{1}{1+\frac{1}{n}}\right|^n = \left(\frac{1}{1+\frac{1}{n}}\right)^n \quad \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{\lim_{n\to\infty}(1+\frac{1}{n})^n} = \frac{1}{e}$

-> So, by the ratio test, since $\frac{1}{e} < 1$, $\sum n!/n^n$ converges
-> So, $n^n$ grows even faster than $n!$

<u>Proof of Ratio test</u>

-> Case: $L < 1$ : We know $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$. So, we can choose $\varepsilon$ and find an $N$ such that $L-\varepsilon < \left|\frac{a_{n+1}}{a_n}\right| < L+\varepsilon$ where $n \geq N$. Since $L<1$ choose $\varepsilon$ such that $L+\varepsilon < 1$. Let $r = L+\varepsilon$, meaning $L < r < 1$. So, $\left|\frac{a_{n+1}}{a_n}\right| < r$ for all $n \geq N$. At $n=N \Rightarrow |a_{n+1}| < r|a_n|$. At $n = N+1 \Rightarrow |a_{n+2}| < r|a_{n+1}| < r^2|a_n|$ At $n = N+k-1 \Rightarrow |a_{n+k}| < r^k|a_n|$. Note that $\sum_{k=1}^{\infty} r^k|a_n|$ converges since $r<1$. By comparison $\sum_{k=1}^{\infty}|a_{n+k}|$ converges. Finally, $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N}|a_n| + \sum_{k=1}^{\infty}|a_{n+k}|$. Since these both converge, it must be that $\sum_{n=1}^{\infty}|a_n|$ converges.

<u>Root Test</u>
-> Given $\sum a_n$, let $L = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}|a_n|^{1/n}$
-> If
① $L<1$ then $\sum a_n$ is absolutely convergent
② $L>1$ then $\sum a_n$ is divergent
③ $L=1$ we don't know, more tests required
-> It is helpful to note that $\lim_{n\to\infty} n^{1/n} = 1$

## Root Test Examples

a) $\sum_{n\geq 1}^{\infty} \frac{n}{2^n}$. We have $\lim_{n\to\infty} \left(\frac{n}{2^n}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{n^{1/n}}{2}\right) = \frac{\lim_{n\to\infty} n^{1/n}}{2} = \frac{1}{2} < 1$

so the series is absolutely convergent

b) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2}$. We have $\lim_{n\to\infty} \left(\left(\frac{n+1}{n}\right)^{n^2}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$, so the series diverges

c) $\sum_{n=1}^{\infty} (-1)^n/\sqrt{n}$. We have $\lim_{n\to\infty} \left(\frac{(-1)^n}{\sqrt{n}}\right)^{1/n} = \lim_{n\to\infty} \frac{(-1)}{n^{1/2n}} = \lim_{n\to\infty} -n^{-\frac{1}{2}n}$

$= \lim -(n^{1/n})^{(-\frac{1}{2})} = -(1)^{(-1/2)} = -\frac{1}{\sqrt{1}} = |-1| \leq 1$, so it is inconclusive

d) $\sum_{n=1}^{\infty} n e^{-n^2}$ we have $\lim_{n\to\infty} (n e^{-n^2})^{1/n} = \lim_{n\to\infty} n e^{-n} = \lim_{n\to\infty} \frac{n}{e^n} = 0 < 1$

-> The root test is a stronger version of the ratio test

-> Whenever the ratio test passes, so will the root test

-> Most of the time, both tests find the same value for L

## Root/Ratio Proof Idea

$\boxed{L<1}$ Choose r such that $L<r<1$. Next, since $|a_n|^{1/n} \to L$ there is an N such that $|a_n|^{1/n} < r$ for $n \geq N \Rightarrow |a_n| < r^n$ for $n \geq N$

Since $\sum r_n$ converges (geometric series with $|r|>1$), $\sum |a_n|$ converges by comparison

$\boxed{L>1}$ Similar, but choose $1<r<L$

$\boxed{L=1}$ Note that $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ have $L=1$ (one converges while another diverges)

Summary of Tests

① Divergence test : $\lim_{n\to\infty} a_n \neq 0$

② P-series test : $1/x^p$ converges for $p > 1$

③ Integral test : corresponding improper integral
   -> W/ remainder estimate

④ Direct comparison test

⑤ Limit comparison test : $\lim_{n\to\infty} \frac{a_n}{b_n} \in (0, \infty)$ for some other $b_n$

⑥ Alternating Series test : $a_n > 0$, $a_{n+1} < a_n$, $\lim_{n\to\infty} a_n = 0$ (for alt. series)
   -> W/ remainder estimate

⑦ Ratio test : $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

⑧ Root test : $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n}$

⑨ Geometric Series (explicit sum) : $\frac{1}{r-1}$

⑩ Telescoping Series : first term plus last term.

Note: $0! = 1$
$0^0 = 1$
(use for most series)

Ex. For what values of $x$ do the following converge

-> Not knowing the value of $x$, we appeal to the ratio test

-> Specifically, we must force $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

a) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n(n+1)}$. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-1)^{n+1}}{2^{n+1}(n+2)} \cdot \frac{2^n(n+1)}{(x-1)^n}\right| = \left|\frac{x-1}{2} \cdot \frac{n+1}{n+2}\right|$

We have $\lim_{n\to\infty} \left|\frac{x-1}{2} \cdot \frac{n+1}{n+2}\right| = \left|\frac{x-1}{2} \cdot 1\right| = \left|\frac{x-1}{2}\right|$ as $n \to \infty$

$\left|\frac{x-1}{2}\right| < 1 \Rightarrow |x-1| < 2 \Rightarrow -1 < x < 3$ in this interval, the series converges

b) $\sum_{n=0}^{\infty} n!(x+2)^n$. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!(x+2)^{n+1}}{n!(x+2)^n}\right| = |(n+1)(x+2)|$

$\lim_{n\to\infty} |(n+1)(x+2)|$ DNE unless $x = -2$, only converges at $x = -2$

c) $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n^n}$. $\left|\frac{a_{n+1}}{a_n}\right| = \cancel{\left|\frac{(x-4)^{n+1}}{(n+1)^{(n+1)}}\right|} = \left|\frac{(x-4)^{n+1}}{(n+1)^{(n+1)}} \cdot \frac{n^n}{(x-4)^n}\right|$

$= \left|(x-4)\left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}\right|$. $\lim_{n\to\infty} \left|(x-4)\left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}\right|$.

$= \left|\lim_{n\to\infty}(x-4)\right| \cdot \left|\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}\right| \cdot \left|\lim_{n\to\infty} \frac{1}{n+1}\right| = \left|(x-4)\left(\frac{1}{e}\right)(0)\right| = 0$

This value is less than 1 for any $x$, so it converges for all $x$

Theorem - Radius of convergence

-> Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. It will either

① Converge for all $x \in \mathbb{R}$ $\qquad R = \infty$

② Converge for $|x-a| < R$ and diverge for $|x-a| > R$ where $0 < R < \infty$

③ Converge only at $x = a$. $\qquad R = 0$

-> The constant $R$ is called the radius of convergence

-> This is how far from the center we can move

-> When $0 < R < \infty$ we must consider the endpoints separately since $a_n \to \infty = 1$ here (so the ratio test doesn't work)

-> We will use another (any other) test in this case

Ex. Continued

-> For $x=3$, the series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n(n+1)} = \sum_{n=0}^{\infty} \frac{2^n}{2^n(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1}$

-> This diverges by the integral test

-> For $x=-1$, we have $\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n(n+1)} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n+1}\right)$

-> This converges by AST

-> So, the series converges at $x \in [-1, 3)$

Interval of Convergence

-> This is the interval over which the power series function converges

-> let I be an interval of convergence for the power series $f(x) = c_n(x-a)^n$, then f is continuous for all $x \in I$

-> I is either

① $(a+R, a+R)$ ③ $[a-R, a+R)$

② $(a-R, a+R]$ ④ $[a-R, a+R]$

Manipulating Power Series

-> Given two power series $\sum b_n(x-a)^n$ and $\sum^{\infty} c_n(x-a)^n$ with radii $R_b$ and $R_c$, it can be shown that $\sum^{\infty}(b \pm c_n)(x-a)^n$ has a radius of at least $\min(R_b, R_c)$

-> Multiplication of a convergent power series by a constant or finite polynomial does not affect convergence or change the radius

-> Ex $\alpha \sum^{\infty} b_n(x-a)^n = \sum \alpha b_n (x-a)^n$

-> Ex. $(x-a)^k \sum^{\infty} b_n (x-a)^n = \sum^{\infty} b_n (x-a)^{n+k}$

-> This is a useful polynomial to multiply by

-> Ex. Consider the series $\sum^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$

-> We have $R=1$ by the ratio test

-> $\sum^{\infty} x^{n+3} = \sum x^3 x^n = x^3 \sum^{\infty} x^n \Rightarrow$ same radius of convergence

-> Note: $\sum^{\infty} x^n$ is a geometric series with $A=1$ and $r=x$.

-> So, it converges to the value $\frac{A}{1-r} = \left(\frac{1}{1-x}\right)$

-> Important result: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \ldots$

-> Only true if $|x| < 1$

-> We can use this to find the sums of... weirder series

## Derivatives and Integrals of power series

-> If $f(x) = \sum_{n=0}^{\infty} b_n (x-a)^n$ has radius $R > 0$

-> $F(x)$ is differentiable on $(a-R, a+R)$

-> $f'(x) \sum_{n=0}^{\infty} n b_n (x-a)^{n-1}$ (basically the power rule)

-> The derivative of the series is equal to the sum of the derivatives of each term ($\frac{d}{dx} \sum a_n = \sum \frac{d}{dx} a_n$)

-> It is also integrable:

$$\int F(x)\,dx = \sum_{n=0}^{\infty} \frac{b_n (x-a)^{n+1}}{n+1} + C$$

-> Integral is the sum of integrals ($\int \sum a_n(x) = \sum \int a_n(x)$)

-> Both series have radius of convergence $R$

## Power Series Representations

-> We can find the sums of series by using $\frac{1}{1-x}$ as a building block

-> Ex. Find a power series representation of $\frac{x}{(1-x)^2}$

-> Let $f(x) = \frac{1}{1-x}$, we know $f(x) = \sum_{x=0}^{\infty} x^n$ for $|x| < 1$.

-> By previous theorem, $f'(x) = \frac{1}{(1-x)^2} = \sum n x^{n-1}$

-> $x f'(x) = x/(1-x)^2 = x \sum n x^{n-1} = \sum^{\infty} n x^n$

-> Using this, find the sum of $\sum_{n=1}^{\infty} \frac{n}{2^n}$

-> We set $x = \frac{1}{2}$, which is within the radius of convergence

-> This gives $\sum_{n=1}^{\infty} n/2^n = \frac{x}{(1-x)^2} = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{(1/2)^2} = 2$

-> Note that this will not work for any $x$ outside of $(a-R, a+R)$. In this case, $(-1, 1)$

-> The endpoints may converge; they must be checked individually

-> Differentiating / Integrating will not change the radius, but it may change the behavior of the endpoints

# Power Series Examples

-> Ex. Use $\frac{1}{1-x}$ to find a series representation for $\ln(1-x)$. What is the interval of convergence

-> The series $\frac{1}{1-x} = \sum^{\infty} x^n$ has interval of convergence $(-1,1)$

-> Note that $\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x}$

-> This suggests integrating the series $\sum x^n$

-> $\int \frac{1}{1-x}dx = \int \sum_{n=0}^{\infty} x^n dx = \sum_{n=0}^{\infty} \int x^n dx$ (for $|x|<1$)

$\Rightarrow -\ln(1-x) + C = \sum^{\infty} x^{n+1}/(n+1)$

-> let $x=0$ to find C. (0 is the center, which makes computation easy)

-> $\ln(1) + C = \sum^{\infty} 0 \Rightarrow C = 0$

-> So, $\ln(1-x) = -\sum_{n=0}^{\infty} x^{n+1}/(n+1)$

-> To find the interval, we must evaluate the endpoints ($x=1$, $x=-1$)

-> $x=1$ : $\sum_{n=0}^{\infty} 1^{n+1}/n+1 = \sum^{\infty} \frac{1}{n+1}$ diverges, by LCT

-> $x=-1$ : $\sum_{n=0}^{\infty} (-1)^{n+1}/(n+1)$ converges by AST

-> So, the interval of convergence is $[-1,1)$

## Substitutions

-> It is possible to replace x with another expression in a power series

-> Must be careful when finding the radius of convergence

## Examples (Cont.)

-> Find the power series representation of $\frac{1}{4+2x}$ centered at 0

-> $\frac{1}{4+2x} = \frac{1}{4(1+\frac{x}{2})} = \frac{1}{4(1-(-\frac{x}{2}))}$. let $u = -\frac{x}{2}$.

-> $\frac{1}{4}\left(\frac{1}{1-u}\right) = \frac{1}{4}\sum_{n=0}^{\infty} u^n$ (for $|u|<1$) $= \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^n}{2^n}\right)$

-> Valid for $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

-> Radius of convergence is $R = 2$.

Examples (cont.)

-> Find a power series for $\arctan(x) = \tan^{-1}(x)$

-> Recall $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} = \frac{1}{1-(x^2)}$. Let $u = -x^2$.

-> $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ for $|u| < 1$.

-> $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$ for $|-x^2| < 1 \Rightarrow |x| < 1$

$= \sum^{\infty} (-1)^n x^{2n}$

-> $\int \frac{1}{1+x^2}\,dx = \arctan(x) = \int \sum^{\infty} (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx$

-> $\arctan(x) + C = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/2n+1$

-> pick $x = 0$ to find $C = 0$

-> So $\arctan(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2n+1}\right) x^{2n+1}(-1)^n = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots \quad |x| < 1$

-> Testing endpoints

$\boxed{x=1}$ we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ which converges by AST

$\boxed{x=-1}$ we get $\sum_{n=0}^{\infty} (-1)^{n+1}/2n+1$ which converges by AST

-> So, the interval of convergence is $[-1, 1]$

-> This suggests $\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Finding Integrals Using Power Series

-> Write $\int_0^1 \arctan(x^4)\,dx$ as a power series

-> $\arctan(u) = \sum^{\infty} (-1)^n u^{2n+1}/(2n+1) \quad |u| < 1$

$\Rightarrow \arctan(x^4) = \sum^{\infty} (-1)^n (x^4)^{2n+1}/(2n+1) = \sum (-1)^n \left(\frac{x^{8n+4}}{2n+1}\right) \quad |x^4| < 1$

integration step skipped

-> $\int_0^1 \arctan(x^4)\,dx \overset{+C}{=} \sum_{n=0}^{\infty} (-1)^n x^{8n+5}\left(\frac{1}{(2n+1)(8n+5)}\right)\Big|_0^1 = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{(2n+1)(8n+5)}\right)$

$= \frac{1}{5} - \frac{1}{39} + \frac{1}{105} - \ldots$

-> $|x^4| < 1 \Rightarrow |x| < 1$

## Taylor Series

-> We found $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$ can be used to find series representations of functions

-> How can we do this for any $f(x)$?

-> Converges to $f(x)$ over some interval.

-> Assume $f(x)$ can be written as a power series centered at $x=a$ and that it converges on the interval $|x-a|<R$ where $R>0$

predicated on the series existing

-> Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2$

-> Note that $f(a) = c_0$

-> $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$

-> $f^{(n)}(x) = \frac{n!}{0!}c_n + \frac{(n+1)!}{1!}c_{n+1}(x-a) + \dots \frac{(n+2)!}{2!}c_{n+2}(x-a)^2 + \dots$

-> $f'(a) = c_1$, $f''(a) = 2c_2$, $f'''(a) = 3!c_3$

-> $f^n(a) = n!c_n \longrightarrow c_n = f^{(n)}(a)/n!$

-> Is the converse true? If we have the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, then is there an $R$ such that $f(x)$ converges for $|x-a|<R$

-> The partial sum of this series is just the $n^{th}$ degree taylor polynomial of $f$ centered at $x=a$

-> What values of $x$ does $\lim_{n\to\infty} T_{n,a}(x) = f(x)$

## Taylor Polynomial Review

-> If $a=0$, it is sometimes called a MacLaurin polynomial

-> $\sin(x)$: $\sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

-> $T_{n,0}(x) = T_{n+1,0}(x)$ when $n$ is odd

-> $\cos(x)$: $\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!}$

-> $T_{n,0}(x) = T_{n+1,0}(x)$ when $n$ is even

-> $e^x$: $T_{n,0}(x) = \sum_{k=0}^{n} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

## Taylor Convergence

-> To determine if $T_{n,a}(x) \to f(x)$ as $n \to \infty$ we look at the difference ~~[illegible]~~ $f(x) - T_{n,a}(x)$

-> Recall: this is $R_{n,a}(x) = f(x) - T_{n,a}(x)$    taylor's theorem

-> Recall: $R_{n,a}(x) = f(x) - T_{n,a}(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$ for some $c$ between $x$ and $a$

-> We want to show: $\lim_{n\to\infty} R_{n,a}(x) = 0$

-> However, remember that it is not actually possible to find $c$

-> Instead, we look for an upper bound on $|f^{(n+1)}(c)|$

$$|R_{n,a}(x) \leq \frac{M}{(n+1)!}(x-a)^{n+1} \quad \text{whenever } |f^{n+1}(c)| \leq M$$

Ex. Let $f(x) = \sin(x)$. Show $T_{n,0}(x) \to \sin(x)$ for all $x \in \mathbb{R}$

-> since $f(x) = \sin(x)$, $|f^{n+1}(x)|$ is either $\cos(x)$ or $\sin(x)$

-> so $|f^{n+1}(c)| \leq 1$ for all $c$

-> So $|R_{n,0}(x)| = \frac{|f^{n+1}(c)|}{(n+1)!}(x-a)^{n+1} \leq \frac{1}{(n+1)!}(x)^{n+1}$

-> Note that $x$ is fixed and $b^n << n!$ for fixed $b$

-> i.e. $\lim_{n\to\infty} b^n/n! = 0$

-> $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \Rightarrow \lim_{n\to\infty} R_{n,0}(x) \leq 0$ (so it is 0)

-> So $\lim_{n\to\infty} T_{n,0}(x) = \sin(x)$, so $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for all $x$

## Taylor Series (formal definition)

-> Given an infinitely differentiable function $f$, the series $\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$ is the Taylor series centered at $x=a$

-> A Taylor series is NOT $f(x)$. It may be

-> $f(x)$ for all $x \in \mathbb{R}$

-> $f(x)$ over an interval $|x-a| < R$

-> only equal to $f(x)$ at $f(a)$

Examples of Taylor Series

-> $f(x) = \ln(x)$ has Taylor series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$

-> $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n}\right| = |x-1| < 1$

-> For all $x \in (0,2)$, we have $\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$

-> Turns out the interval of convergence is $(0, 2]$

-> $\ln(x)$ cannot equal its taylor series outside of this interval

-> Consider $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$ / this is just the limit, this function is continuous

-> This can be shown to satisfy $f^{(n)}(0) = 0$ for all $n \geq 0$

-> (done using first principles)

-> So, the taylor series centered at 0 is the constant function 0

-> This converges for all $x$, BUT IS NOT $f(x)$ !!

-> Just because the Taylor series converges does not mean it matches with the function all all points

Binomial Series

-> The function $(x+1)^b$, $b \in \mathbb{R}$ has derivatives

-> $b(x-1)^{b-1}$ (first derivative)

-> $f''(x) = b(b-1)(x+1)^{b-2}$

-> $f^{(n)}(x) = b(b-1)(b-2)\ldots(b-(n-1))(x+1)^{b-n} = \frac{b!}{(b(n-1))!}(x-1)^{b-n}$

-> So, the Taylor series is (centered at $x=0$)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = 1 + bx + \frac{b(b-1)}{2}x^2 + \ldots + \frac{b(b-1)\ldots(b-(n-1))x^n}{n!} + \ldots$$

-> We define $\binom{b}{n} = {}_bC_n = \frac{b(b-1)\ldots(b-(n-1))}{n!} =$

as the binomial coefficient

-> if $b \in \mathbb{N}$, we have $\binom{b}{n} = \frac{b!}{(b-n)!\,n!}$ for $b \geq n$, 0 otherwise

-> otherwise: $\binom{5.7}{3} = \frac{(5.7)(4.7)(3.7)}{3!}$

-> More concisely, the Taylor series is $\sum_{n=0}^{\infty} \binom{b}{n} x^n$

-> For which $x$ does this converge

-> Will it converge to $f(x)$ for the points above

<u>Convergence of Binomial Series</u>

-> We will use the ratio test

-> $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\binom{b}{n+1} x^{n+1}}{\binom{b}{n} x^n}\right| = \left|\frac{\binom{b}{n+1}}{\binom{b}{n}} x\right| = |x| \left|\frac{b-n}{n+1}\right|$

$\lim_{n \to \infty} |x| \left|\frac{b-n}{n+1}\right| = \lim_{n \to \infty} |x| \left|\frac{b-n}{n+1}\right| = |x| < 1$

-> So, it will converge when $|x| < 1 \Rightarrow x \in (-1, 1)$

-> There is not an easy way to check the endpoints, but it turns out it is

-> $[-1, 1]$ when $b \geq 0$, $b \in \mathbb{N}$

-> $(-1, 1]$ when $b \in (-1, 0)$

-> $(-1, 1)$ when $b \leq -1$

-> $(-\infty, \infty)$ when $b \in \mathbb{N}$ (reduces to a finite polynomial)

-> What does $\sum_{n=0}^{\infty} \binom{b}{n} x^n$ converge to?

-> Assume $|x| < 1$, let $f(x) = \sum_{n=0}^{\infty} \binom{b}{n} x^n$

-> Note: $(n+1)\binom{b}{n+1} = \binom{b}{n}(b-n)$

-> We see that $f'(x) = \sum_{x=1}^{\infty} n \binom{b}{n} x^{n-1} = \sum_{x=0}^{\infty} (n+1)\binom{b}{n+1} x^n$

if the bottom index is 0, we just get 0, which doesn't give a useful derivative

$= \sum_{n=0}^{\infty} (b-n)\binom{b}{n} x^n$ (from aside) $= b \underbrace{\sum_{n=0}^{\infty} \binom{b}{n} x^n}_{\text{this is } f(x) \text{ !!!}} - \underbrace{\sum_{n=0}^{\infty} n \binom{b}{n} x^n}_{f'(x) \text{ times } x \text{ !!!}}$

$= b f(x) - x \sum_{n=0}^{\infty} n \binom{b}{n} x^{n-1} = b f(x) - x f'(x) = f'(x)$

-> This is a differential equation!

-> $f'(x) = \frac{b f(x)}{1+x} \Rightarrow \frac{1}{f} df = b \frac{1}{1+x} dx \Rightarrow \int \frac{1}{f} df = b \int \frac{1}{1+x} dx$

$\Rightarrow \ln|f| = b \ln|1+x| + C \Rightarrow \ln|f| = \ln(1+x)^b + C$ since $|x| < 1$

$\Rightarrow f(x) = A(1+x)^b$, $A = \pm e^C$. Recall $f(x) = \sum \binom{b}{n} x^n$, so $f(0) = 1$

$\Rightarrow A(1+0)^b = 1 \Rightarrow A = 1$. So, $f(x) = (1+x)^b = \sum_{n=0}^{\infty} \binom{b}{n} x^n$ for $|x| < 1$

## Applications of Taylor Series

-> Finding series representations of irrational numbers

-> Calculating sums that are not geometric / telescoping

-> Approximating hard integrals

-> Evaluating limits

## Example: Series for $\pi$

-> Recall: $(1+x)^b = \sum^{\infty} \binom{b}{n} x^n$ when $|x|<1$

-> Let $b = -\frac{1}{2}$ and sub $x \to -x^2$

-> $(1-x^2)^{-1/2} = \sum^{\infty} \binom{-1/2}{n}(-x^2)^n \Rightarrow \frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n}$ for $|-x^2|<1 \Rightarrow |x|<1$

-> We can then integrate the power series

-> $\int_0^x \frac{1}{\sqrt{1-t^2}}\, dt = \int_0^x \sum_{n=0}^{\infty} \binom{-1/2}{n}(-1)^n t^{2n}\, dt \Rightarrow$

-> $\arcsin(x) - \arcsin(0) = \sum^{\infty} \binom{-1/2}{n} (-1)^n \left(x^{2n+1}/2n+1\right) \Rightarrow$

-> $\arcsin(x) = \sum^{\infty} (-1)^n \binom{-1/2}{n} \frac{x^{2n+1}}{2n+1}$

-> When $x = \frac{1}{2}$: $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} = \sum^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \frac{1}{2^{2n+1}(2n+1)}$

$\Rightarrow \pi = 3 \sum^{\infty} \binom{-1/2}{n}(-1)^n \left(1/4^n(2n+1)\right)$

## Examples: Sums of Series

$\frac{1}{1-x}$ if no factorials

-> Look for similarities to $\frac{1}{1-x}$, $e^x$, $\sin(x)$, $\cos(x)$, $(1+x)^b$

-> Example: $\sum^{\infty} \frac{n^3}{e^n}$

-> $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \left(\frac{d}{dx}\right) \Rightarrow \frac{1}{(1-x)^2} = \sum^{\infty} n x^{n-1} \Rightarrow$ (times $x$)

$\Rightarrow x/(1-x)^2 = \sum^{\infty} n x^n \Rightarrow \left(\frac{d}{dx}\right) \Rightarrow (1+x)/(1-x)^3 = \sum^{\infty} n^2 x^{n-1}$

$\Rightarrow$ (times $x$) $\Rightarrow x(1+x)/(1-x)^3 = \sum^{\infty} n^2 x^n \Rightarrow \left(\frac{d}{dx}\right.$ and $\left.x\right)$

$\Rightarrow \frac{x^3+4x^2+x}{(1-x)^4} = \sum_{n=1}^{\infty} n^3 x^n$

-> Now, let $x = 1/e \Rightarrow \frac{e+4e^2+e^3}{(e-1)^4} = \sum_{n=1}^{\infty} \frac{n^3}{e^n}$

-> Example: $\sum_{n=2}^{\infty} \frac{(-1)^n \pi^{2n-4}}{(2n-1)!}$

-> Recall: $\sin(x) = \sum^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$

-> $\frac{1}{\pi^3} \sum_{n=2}^{\infty} \frac{(-1)^n \pi^{2n-1}}{(2n-1)!} = \frac{1}{\pi^3}\left[\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n-1}}{(2n-1)!}}_{-\sin(\pi)} - (-\pi)\right]$ term for $n=1$

$= \frac{1}{\pi^3}[0 + \pi] = \frac{1}{\pi^2}$

## More Applications

-> Let $f(x) = \arctan(x^3)$. Find $f^{(15)}(0)$, $f^{(17)}(0)$

-> Recall $g(u) = \arctan(u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n+1}}{2n+1}$, $|u|<1$

-> For converging power series, we must have $c_k = \frac{f^{(k)}(0)}{k!}$

-> $f(x) = \arctan(x^3) = \sum \frac{(-1)^n (x^3)^{2n+1}}{2n+1} = \sum \frac{(-1)x^{6n+3}}{2n+1}$

-> $k = 15 = 6n+3 \Rightarrow k = 2 \Rightarrow \frac{f^{(15)}(0)}{15!} = \frac{(-1)^2}{5} \Rightarrow f^{(15)}(0) = \frac{15!}{5}$

-> $k = 17 \Rightarrow$ no integer for $n$. There is no $f^{(17)}$ term, so it is 0

-> We are simply matching the coefficients of terms in the series

-> Find a series representation of $f'(x) = \int_0^x \cos(t^2)\,dx$

-> Note: this function does not have an elementary antiderivative

-> Recall $\cos(u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n)!}$

-> So, $\cos(u^2) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n}}{(2n)!} \Rightarrow \int \cos(t^2)\,dt = \sum_{n=0}^{\infty} \left[ \frac{(-1)^n t^{4n+1}}{(4n+1)(2n)!} \Big|_0^x \right]$

$$= \sum \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}$$

-> Compute $\lim_{x \to 0} \frac{1+3x^2\ln(1+x^2) - e^{x^4}}{x\sin(x^3)}$

-> This is $\frac{0}{0}$, but using hopital's rule would take 4 derivations

-> Recall $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4}$$

$$\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \dots$$

-> $\frac{1+3x^2\ln(1+x) - e^{x^4}}{x\sin(x^3)} = \frac{1+3x^2\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} + \dots\right) - \left(1 + x^4 + \frac{x^8}{2} + \dots\right)}{x\left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} + \dots\right)}$

$$= \frac{3x^4 - \frac{3x^6}{2} + x^8 - x^4 - \frac{x^8}{2} + \dots}{x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!}} = \frac{3 - 3\frac{x^2}{2} + x^4 - 1 - \frac{x^4}{2}}{1 - \frac{x^4}{3!} + \frac{x^{12}}{5!}}$$

-> As $x \to 0$, we get $\frac{3-1}{1} = \frac{2}{1} = 2$

-> We can use big-O for the higher-order terms (...) notation