1.1 Absolute Values

defined as $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ for $x \in \mathbb{R}$ $\boxed{|x| = |-x|}$

application: distance between two points a and b $= |a-b| = |b-a|$

1.1.1 Triangle inequalities

Triangle inequality 1: $|x-y| \leq |x-z| + |z-y|$ $x, y, z \in \mathbb{R}$
"the sum of two sides of a triangle is larger than the remaining one"
"distance between two points is a straight line between them"

Triangle inequality 2: $|x+y| \leq |x| + |y|$ $x, y \in \mathbb{R}$
derived from first triangle inequality

Inequalities in other forms

$|x| \leq \delta \quad = \quad -\delta \leq x \leq \delta \quad = \quad x \in (-\delta, \delta)$

$|x-a| \leq \delta \quad = \quad a-\delta \leq x \leq \delta + a \quad = \quad x \in (a-\delta, a+\delta)$
a is the offset from the origin

$0 < |x-a| < \delta = (a-\delta, a+\delta) \setminus \{a\} = x \in ((a-\delta, a) \cup (a, a+\delta))$
$\{a\}$ means excluding $x = a$
This basically says $|x-a| \neq 0$, otherwise same as $|x-a| < \delta$

Solving inequalities w/ two parts:
-> solve each side of the inequality
-> the solution is the overlap of the two sets / ranges

1.2.1 Introduction to sequences

Sequence: an ordered list (may be finite or infinite)

-> Notation 1: $a_1, a_2, a_3 \dots a_n \dots$ ($a \in S$)

-> Notation 2: $\{a_1, a_2, a_3 \dots a_n \dots\}$ $\{a_n\}_{n=1}^{\infty}$

May be defined explicitly $a_n = 2n + 5$

May be defined recursively $a_1 = 7,\ a_n = 2a_{n-1}$

Sequences are discrete representations of functions

1.2.3 Subsequences and tails

Subsequence: if $\{a_n\}$ is a sequence and $\{n_k\} \in \mathbb{N}$ is a strictly ascending sequence, a subsequence is a third sequence in the form $\{a_{n_k}\} = \{a_{n_1}, a_{n_2}, a_{n_3}, \dots a_{n_k} \dots\}$

Tail: subsequence $\{a_k, a_{k+1}, a_{k+2} \dots\}$

Subsequence containing every term of original sequence after given element

1.2.4 Limits of Sequences

Convergent sequence: Sequence has a limit L that the terms approach (can be arbitrarily close to L)

Limit: $L \in \mathbb{R}$ is the limit of $\{a_n\}$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that if $n \geq N$ then $|a_n - L| < \varepsilon$

-> ε is a "tolerance" that can be made arbitrarily small

-> $a_n \in (L - \varepsilon, L + \varepsilon)$

We can write $|a_n - L| < \varepsilon$ in terms of ε to prove a limit

-> ex. $a_n = 1/\sqrt{n}$ (limit = 0) -> $|1/\sqrt{n} - 0| < \varepsilon$ ->

$1/\sqrt{n} < \varepsilon$ -> $\sqrt{n} > \frac{1}{\varepsilon}$ -> $n > \frac{1}{\varepsilon^2}$ -> we know we can always find an n that satisfies th…

-> we can write n in terms of ε

-> meaning: how many terms do we need before we are within ε?

Formal definition 2: $\lim_{n\to\infty} a_n = L$ if for any $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains a tail of $\{a_n\}$

All of the following statements are equivalent:

I) $\lim_{n\to\infty} a_n = L$

II) for all $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains a tail of $\{a_n\}$

III) for all $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains infinitely many terms of $\{a_n\}$

IV) Every interval containing L contains a tail of $\{a_n\}$

V) Every interval containing L contains all but infinitely many terms of $\{a_n\}$

A sequence cannot have two limits; an infinite sequence that doesn't have one limit is said to diverge

-> Ex. $\{(-1)^n\}_{n=1}^{\infty}$ is a divergent sequence

-> Proof. If 1 (or -1) is a "limit", a tail could be formed within the interval $(0.9, 1.1)$, but it can't

-> Theorem: if a series has a limit L, L is unique

If a sequence only contains non-negative numbers, its limit cannot be negative $((\forall a_n \text{ in } \{a_n\} \geq 0) \Rightarrow (L \geq 0))$

More generally, if all of the terms of a sequence are within α and β, the limit L must also be within α and β $(\alpha \leq a_n \leq \beta \Rightarrow \alpha \leq L \leq \beta)$

1.2.5

A sequence diverges if $\lim_{n\to\infty} a_n = \infty$.

-> for all $M > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $a_n > M$

-> A sequence diverges if every interval in the form (M, ∞) contains a tail

-> Opposite signs for divergence to $-\infty$

$$\left.\begin{array}{l} \text{If } \alpha > 0,\ \lim_{n\to\infty} n^{\alpha} = \infty \\ \\ \text{If } \alpha < 0,\ \lim_{n\to\infty} n^{\alpha} = 0 \end{array}\right] \text{ limits of exponents with bases} \to \infty$$

Proving recursive sequences

We can use the MCT and mathematical induction to prove the limits of recursive sequences

-> Induction: proof technique that allows us to prove infinitely many related statements
  -> 1. Prove $P(n)$ for some n
  -> 2. Prove $P(n) \Rightarrow P(n+1)$

1. Prove that the sequence is monotonic
2. Prove that the sequence is bounded above / below
3. By the MCT, the sequence must converge
4. Find the limit using $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$

Induction proof example: $a_1 = 1$, $a_{n+1} = (1+a_n)/3$

1. Claim: $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. Proof: Base case: $n=1$, $1 \geq \frac{2}{3}$ $\square$. Inductive step: Suppose $a_k \geq a_{k+1}$ for some $k \in \mathbb{N}$ (in this case $k=1$). Then we have $1+a_k \geq 1+a_{k+1}$ then $(1+a_k)/3 \geq (1+a_{k+1})/3$, which is just $a_{k+2} \geq a_{k+1}$. Therefore, $\{a_n\}$ is non-increasing.

2. Claim: $\forall n \in \mathbb{N}$, $a_n \geq \frac{1}{3}$. Proof: Base case: $1 \geq \frac{1}{3}$. Inductive step: Suppose $a_k \geq \frac{1}{3}$ for some $k \in \mathbb{N}$. Since $a_k \geq \frac{1}{3}$, we have $1+a_k \geq 1+\frac{1}{3} = \frac{4}{3}$. So, $a_{k+1} = (1+a_k)/3 \geq \frac{4}{9} \geq \frac{3}{9} = \frac{1}{3}$. Thus, $a_n \geq \frac{1}{3}$ for all $n \in \mathbb{N}$; $\{a_n\}$ is bounded below

3. $\{a_n\}$ is non-increasing and bonded below, so by the MCT it must converge

4. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L \rightarrow L = \frac{1+L}{3} \rightarrow 3L = 1+L \rightarrow$ $2L = 1$, $L = \frac{1}{2}$

Thus, the limit of $\{a_1 = 1, a_k = \frac{1+a_n}{3}\}$ is $\frac{1}{2}$ $\square$

2.1 Limits for Functions

Heuristic definition: $f(x)$ has a limit L if as x gets closer and closer to a, $f(x)$ gets closer and closer to L.

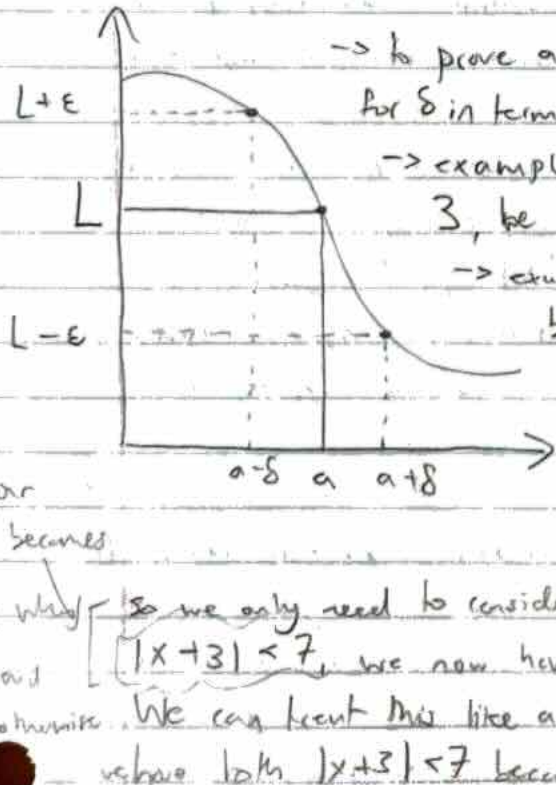
-> Not precise, we need something like ε

-> Because we're now continuous in two dimensions, we need δ as well

Formal definition: Let f be a function and let $a \in \mathbb{R}$. f has a limit L as x approaches a if for any positive tolerance $\varepsilon > 0$, we can find a cutoff distance $\delta > 0$ such that if the distance from x to a is less than δ, and if $x \neq a$, then $f(x)$ approximates L with an error less than ε.

Symbolic definition: If $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

Notation: $\lim_{x \to a} f(x) = L$, or $x \to a$ and $f(x) \to L$ (for shorthand)



-> to prove a limit with the formal definition, find an expression for δ in terms of ε in the form $\delta \leq f(\varepsilon)$

-> example: show $\lim_{x \to 2} 3x+1 = 7$. Because the slope is 3, we know that $3 \cdot \delta \leq \varepsilon$, so $\delta \leq \frac{\varepsilon}{3}$

-> example: show that $\lim_{x \to 3} x^2 = 9$: We can't use $\delta = \frac{\varepsilon}{x+3}$ because $x+3$ isn't constant, but we have a trick: if we find a δ that works for a particular ε, any smaller δ will also satisfy the definition of the limit for the same ε. We can always assume that $\delta \leq 1$, so we have $0 < |x-3| < \delta \leq 1$, so we only need to consider: $2 < x < 4$. Because $x < 4$, we know $|x+3| < 7$, we now have $|x^2 - 9| = |(x-3)(x+3)| = |x+3||x-3| < 7|x-3|$. We can treat this like a linear case: $\delta = \min\{1, \frac{\varepsilon}{7}\}$. Anyway, when $0 < |x-3| < \delta$, we have both $|x+3| < 7$ because $\delta \leq 1$ and $|x+3| < \frac{\varepsilon}{7}$. It follows that if $0 < |x-3| < \delta$, then $|x^2 - 9| = |x-3||x+3| < 7|x-3| = 7\left(\frac{\varepsilon}{7}\right) = \varepsilon$.

now, our $\delta \leq \frac{\varepsilon}{x+3}$ becomes $= \frac{\varepsilon}{7}$ when $\delta \leq 1$ and $\delta = 1$ otherwise

-> strategy: work backward from $|f(x) - L|$ to get to $|x - a|$

ex. -> $f(x) = 5x - 3$, prove that $\lim_{x \to 1} (5x - 3) = 2$.

-> $|f(x) - 2| < \varepsilon$ -> $|5x - 3 - 2| < \varepsilon$ -> $|5x - 5| < \varepsilon$

-> $5|x - 1| < \varepsilon$ -> $|x - 1| < \frac{\varepsilon}{5}$ (for any value of ε, $\delta = \frac{\varepsilon}{5}$ makes the limit)

-> strategy: use the triangle inequality

-> $g(x) = x^2 + 1$, prove $\lim_{x \to 7} (x^2 + 1) = 50$

-> $|x^2 + 1 - 50| = |x^2 - 49| = |x + 7||x - 7| < \delta |x + 7|$ ($x - a$)

-> Assuming $|x - 7| < 1$ (which is fine, we can show the limit must be within 1 unit of $|x - 7|$ using a specific ε), we have $|x| < 8$.

-> By the triangle inequality, we have $|x + 7| < |x| + |7| = 15$, meaning $|x + 7| < 15$.

-> If $|x + 7| < 15$ when $|x - 7| < 1$ (basically when we're near the limit on the number line), we can set δ as $\min(1, \frac{\varepsilon}{15})$, where it will always be smaller than ε (because we saw $|f(x) - L| < \delta |x + 7|$ and $|x + 7| < 15$.

Sometimes, there will be points on a function that don't have a limit, or seem to have two limits (one from each side). In those cases, the limit doesn't exist.

-> Ex. $f(x) = \frac{|x|}{x}$, $\lim_{x \to 0} f(x)$ doesn't exist. We can prove this by showing that the limit must be inside 2 disjoint intervals

-> If $\varepsilon = \frac{1}{2}$, we can show that if $|x| < \frac{1}{2}$, the limit must be with $\frac{1}{2}$ a unit of $f(x)$ on either side (up or down). However, the two sides are 2 units apart, so the limit can't exist.

Remarks on the existence of limits:

-> For $\lim_{x \to a} f(x)$ to exist, $f(x)$ must be defined on an open interval (α, β) containing $x = a$, except possibly at $x = a$

-> The value of $f(a)$, if it's defined at all, does not affect the existence of the limit or its value

-> If two functions are equal, except possibly at $x = a$, their limiting behavior at a is identical

2.2 Sequential Characterization of Limits

Let $f(x)$ be defined on an open interval containing $x=a$, except possibly at $x=a$. Then, the two following statements are equivalent

I) $\lim_{x \to a} f(x)$ exists and is equal to L.

II) If $\{x_n\}$ is a sequence with $x \neq a$ and $x_n \to a$, then $\lim_{n \to \infty} f(x_n) = L$.

Assume that $\lim_{x \to a} f(x) = L$ and that $\lim_{x \to a} f(x) = M$. Then $L = M$.
(limits are unique for functions)

* most of our rules about sequences can be ported over to functions in this way

Strategy for showing a limit doesn't exist:
-> Option 1: Find a sequence $\{x_n\}$ with $x_n \to a$, $x_n \neq a$ for which $\lim_{n \to \infty} f(x)$ does not exist.
-> Option 2: Find two sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to a$ and $y_n \to a$ and $x, y \neq a$ for which $\lim_{n \to \infty} f(x_n) = L$ and $\lim_{n \to \infty} f(y_n) = M$, but $L \neq M$

examples:

2.3 Arithmetic Rules for Limits of Functions

Theorem: let $f(x)$ and $g(x)$ be functions and let $a \in \mathbb{R}$. Assume that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then:

I) Assume that $f(x) = c$ for every $x \in \mathbb{R}$. Then $\lim_{x\to a} f(x) = c$

II) For any $c \in \mathbb{R}$, $\lim_{x\to a} c \cdot f(x) = cL$

III) $\lim_{x\to a} f(x) + g(x) = L + M$

IV) $\lim_{x\to a} f(x)g(x) = LM$

V) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ as long as $M \neq 0$

VI) $\lim_{x\to a} (f(x))^\alpha = L^\alpha$ for all $\alpha > 0$, $L > 0$

VII) Assume that $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and $\lim_{x\to a} g(x) = 0$. Then $\lim_{x\to a} f(x) = 0$.

Theorem: if $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$

2.4 One-sided limits

Let $f(x)$ be a function and let $a \in \mathbb{R}$. We say $f(x) \to L$ from the right ($\lim_{x\to a^+} f(x) = L$) if for any $\epsilon > 0$, we can find a $\delta > 0$ such that if $0 < x-a < \delta$, and $x > a$, then $|f(x) - L| < \epsilon$.

the limit from the left is defined oppositely ($\lim_{x\to a^-} f(x) = L$) $0 < a-x < \delta$

If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$ (and the contrapositive is also true: if $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, then $\lim_{x\to a} f(x)$ does not exist)

The squeeze theorem (2.5) – for functions!

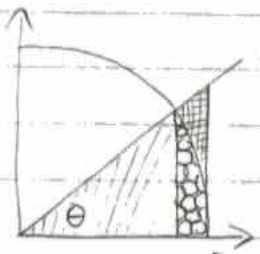
If we have three functions $f(x)$, $g(x)$, and $h(x)$ that are continuous over some interval I (except possibly a) such that $f(x) \le g(x) \le h(x)$ for $x \in I$,

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \implies \lim_{x\to a} g(x) = L$$

We can use the squeeze theorem to show that $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$ by setting $f(x) = -x$ and $h(x) = x$

2.6: The fundamental trigonometric limit $\boxed{\lim_{x\to 0} \frac{\sin(x)}{x} = 1}$

Proof: consider $0 < \theta < \frac{\pi}{2}$.



unit circle $0<\theta<\frac{\pi}{2}$

We know geometrically that < + < + + (by area). By the area of a triangle, we have

$= \frac{1}{2}\cos(\theta)\sin(\theta)$ (from $A = bh$)

By the properties of arcs and radians, we have

$=$ angle * area of circle $= \frac{\theta}{2\pi} \cdot \pi = \frac{\theta}{2}$

And again by trigonometric properties, we have $= \frac{1}{2}\tan(\theta)$

So: $\frac{1}{2}\cos(\theta)\sin(\theta) \le \frac{\theta}{2} \le \frac{1}{2}\tan\theta \implies \cos(\theta)\sin(\theta) \le \theta \le \tan\theta$

times $\frac{1}{\sin(\theta)}$

$$\implies \cos(\theta) \le \frac{\theta}{\sin(\theta)} \le \frac{1}{\cos(\theta)} \implies \frac{1}{\cos(\theta)} \ge \frac{\sin(\theta)}{\theta} \ge \cos(\theta).$$

Because we know $\lim_{x\to 0} \frac{1}{\cos(x)} = 1$ and $\lim_{x\to 0} \cos(x) = 1$, by the squeeze theorem, it must be that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

This limit is very useful for computing trig limits

2.7 : Limits at ∞ for functions

sequence definition basically

We say $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$, we can find a cutoff $N > 0$ such that if $x > N$, then $f(x)$ approximates L with an error less than ε. ($|f(x) - L| < \varepsilon$)

Limits at $-\infty$ can be defined similarly

If $\lim_{x\to\pm\infty} f(x) = L$, we say L is a horizontal asymptote of $f(x)$

If a limit is ∞ at ∞, then for every $M > 0$, there exists a cutoff $N > 0$ such that if $x > N$, $f(x) > M$.
-> In this case $\lim_{x\to\pm\infty} f(x) = \pm\infty$, or $\lim_{x\to\pm\infty} f(x)$ doesn't exist

The squeeze theorem also holds for limits to $\pm\infty$

2.7.2 : Fundamental log limit $\boxed{\lim_{x\to\infty} \frac{\ln(x)}{x} = 0}$

Proof: we know $\ln(u) < u$ for all $u > 0$. Also, we know $u \geq 1 \Rightarrow \ln(u) \geq 0$
So, because $x^{\frac{1}{2}}x^{\frac{1}{2}} = x$, we have

$$0 \leq \frac{\ln(x)}{x} = \frac{\ln(x^{\frac{1}{2}}x^{\frac{1}{2}})}{x} = \frac{2\ln(x^{\frac{1}{2}})}{x^{\frac{1}{2}}} = \frac{2}{x^{\frac{1}{2}}} \cdot \frac{\ln(x^{\frac{1}{2}})}{x^{\frac{1}{2}}} \leq \frac{2}{x^{\frac{1}{2}}}$$, so, we have

$$0 \leq \frac{\ln(x)}{x} \leq \frac{2}{x^{\frac{1}{2}}} \Rightarrow \lim_{x\to\infty} 0 \leq \lim_{x\to\infty} \frac{\ln(x)}{x} \leq \lim_{x\to\infty} \frac{2}{x^{\frac{1}{2}}} \Rightarrow 0 \leq \lim_{x\to\infty} \frac{\ln(x)}{x} \leq 0$$

Therefore, $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$

From there, we can show that any exponential function x^p where $p > 0$ eventually surpasses $\ln(x)$

2.7.3 Vertical Asymptotes and infinite limits

limits equal to ∞ (i.e. $\lim_{x \to 0} \frac{1}{x}$) may have different behavior when approached from the right (above) or the left (below). Specifically, a right-hand limit may equal ∞, while the left limit equals $-\infty$ (like $\lim_{x \to 0} \frac{1}{x}$)

Formal expression:

$f(x)$ has a limit of ∞ from above as $x \to a$ if, for every cutoff $M > 0$, we can find a cutoff $\delta > 0$ such that if $x > a$ and $|x - a| < \delta$, then $f(x) > M$. That is, $a < x < a + \delta \Rightarrow f(x) > M$. This is expressed as $\lim_{x \to a^+} f(x) = \infty$.

There are corresponding definitions for left-hand limits and limits at $-\infty$

If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \infty$, we say $\lim_{x \to a} = \infty$

If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = -\infty$, we say $\lim_{x \to a} f(x) = -\infty$

vertical asymptote too?

If $\lim_{x \to a^\pm} f(x) = \pm\infty$ and $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, we say $f(x)$ has a vertical asymptote at $x = a$

To determine the sign of ∞, show which intervals over which the function is positive and/or negative, and choose accordingly

2.8 Continuity

A function is continuous at $x = a$ if:

$\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = f(a)$

otherwise, we say $f(x)$ is discontinuous at $x = a$. If a function is continuous over an interval, this must be true for every value in that interval.

Second formal definition: $f(x)$ is continuous at $x = a$ if for every $\varepsilon > 0$, there exists a cutoff distance $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Sequential characterization: $f(x)$ is continuous at $x = a$ iff when $\lim_{x \to \infty} \{x_n\} = a$, $\lim_{x \to \infty} f(x_n) = f(a)$

$\boxed{\text{Also: } f(x) \text{ is continuous at } x = a \text{ if } \lim_{h \to 0} f(a + h) = f(a)}$

2.8.1 Types of discontinuities

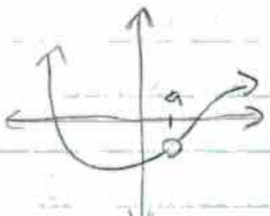
Our first definition of continuity has two requirements:

(i) $\lim_{x \to a} f(x)$ exists

(ii) $\lim_{x \to a} f(x) = f(a)$

If a function is discontinuous, either (ii) fails or (i) fails (and so too does (ii))

Case I: (i) is true, but (ii) is false: the "correct limit" exists, but it is undefined:



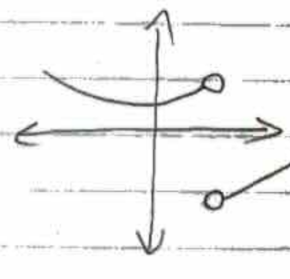
In this case, there is a "hole" in the graph at $x = a$. This is called a "removable discontinuity" because it can be "fixed" trivially by adding the point to the definition of $f(x)$.

Example: $f(x)\ \frac{x^2-1}{x+1}$ at $x=-1$ (rest of function acts as $f(x) = x-1$)

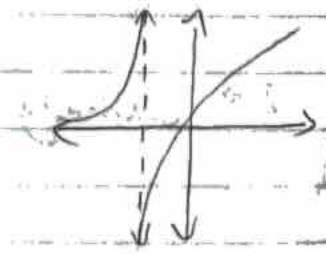
Case II: Essential discontinuity: can't be easily "patched"

Type I: Finite jump discontinuity: $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist, but aren't equal: there is a jump of length $\left|\lim_{x \to a^+} f(x) - \lim_{x \to a^-} f(x)\right|$ between the two sides of a

$\boxed{ex.\ \lim_{x \to 0} \frac{|x|}{x}}$

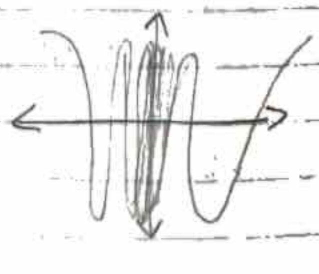


Type 2: Infinite jump: at least one of $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ is equal to $\pm\infty$, so $\left|\lim_{x \to a^+} f(x) - \lim_{x \to a^-} f(x)\right| = \infty$, hence it being an infinite jump. Vertical asymptotes are these type



Type 3: Oscillatory discontinuity: $f(x)$ is bounded near $x = a$, but doesn't have a limit because of an infinite number of oscillations near $x = a$. There is no obvious break, but $f(a)$ is still undefined. Often happens when the argument of a trig function is ∞

$\boxed{ex.\ \lim_{x \to 0} \sin\left(\frac{1}{x}\right)}$



2.8.2 Continuity of common functions

Polynomial functions $p(x)$ are continuous at every point $a \in \mathbb{R}$

$\sin(x)$, $\cos(x)$ are continuous at every point $a \in \mathbb{R}$:

$\lim_{x \to 0} \sin(x) = 0 = \sin(0)$ and $\lim_{x \to 0} \cos(0) = 1 = \cos(0)$, so

$\lim_{x \to a} \sin(x) = \lim_{h \to 0} \sin(a+h) = \lim_{h \to 0} \sin(a)\cos(h) + \sin(h)\cos(a) = \sin(a) \cdot 1 + 0 \cdot \cos(a) = \sin(a)$

{similar proof for cos}

continuous $x > 0$

e^x and $\boxed{\ln(x)}$ are continuous at every point $a \in \mathbb{R}$:

e^x is continuous at $x = 0$ ($\lim_{h \to 0} e^h = e^0 = 1$), so

$\lim_{x \to a} e^x = \lim_{h \to 0} e^{a+h} = \lim_{h \to 0} e^a e^h = e^a \lim_{h \to 0} e^h = e^a \cdot 1 = e^a$

that's their inverse

Because $\ln(x)$ is $f^{-1}(x)$ if $f(x) = e^x$, we know that it is a reflection of $y = e^x$ over the line $y = x$, so it must also be continuous.

Continuity of inverses: assume $y = f(x)$ is invertible, and $x = g(y)$. If $f(x)$ is continuous at a and $f(a) = b$, then g is continuous at $y = b = f(a)$.

Arithmetic rules for continuous functions

If $f(x)$ and $g(x)$ are continuous at $x = a$, then:

I) $f(x) + g(x)$ is continuous at $x = a$

II) $f(x)g(x)$ is continuous at $x = a$

III) If $g(a) \neq 0$, then $\frac{f(x)}{g(x)}$ is continuous at $x = a$ (converse is true)

If $f(x)$ is continuous at $x = a$ and $g(x)$ is continuous at $x = f(a)$, then

I) $(g \circ f)(x) = g(f(x))$ is continuous at $x = a$

(this can be shown using sequential characterization)

2.8.4 Continuity on an interval

If $f(x)$ is continuous over (a,b), it must be continuous at each $x \in (a,b)$

If $f(x)$ is continuous, it is continuous at each $x \in \mathbb{R}$

A function $f(x)$ is continuous over $[a,b]$ if

I) It is continuous at each $x \in (a,b)$

II) $\lim_{x \to a^+} f(x) = f(a)$ III) $\lim_{x \to b^-} f(x) = f(b)$

for example: $\sqrt{1-x^2}$ is continuous over $x \in [-1,1]$, not just $(-1,1)$

2.9 Intermediate Value Theorem

Assume that $f(x)$ is continuous over the closed interval $[a,b]$ and either

$f(a) < \alpha < f(b)$ or $f(a) > \alpha > f(b)$

Then, there exists a $c \in (a,b)$ such that $f(c) = \alpha$

2.9.1 Approximate solutions of equations

bisection method

The IVT proves that a point α must exist, but not how to find it. However, we can use a binary search algorithm with the IVT to locate the 0s of a function, for example

-> If there is a sign change between two points, there must be a 0 by the IVT

-> If we want to find points where $f(x) = n$, calculate the 0s of $f(x) - n$ instead

-> an error bound ε can be defined; stop if the distance between bounds is less than ε

2.10 Extreme Value Theorem

Global Maxima and Minima: suppose I is an interval, and $f: I \to \mathbb{R}$

aka "absolute"

c is a global maximum of $f(x)$ on I if $c \in I$ and $f(x) \leq f(c)$ for all $x \in I$

c is a global minimum of $f(x)$ on I if $c \in I$ and $f(x) \geq f(c)$ for all $x \in I$

c is a global extremum of $f(x)$ on I if it is either a global maximum or minimum for $f(x)$ on I

A function may not have a global max or min if it is defined on an open interval

ex. $x \in (0,1)$, $f(x) = x$ has neither a global max or min

$x \in [0,1]$, $f(x) = x$ does though (1 and 0 respectively)

$x \in (0,1)$, $f(x) = 1 - x^2$ has a max at $x = \frac{1}{2}$, but no min

Extreme value theorem: if $f(x)$ is continuous over $[a,b]$, then there exist c_1 and c_2 such that

$$f(c_1) \leq f(x) \leq f(c_2)$$

for all $x \in [a,b]$

using f(x) as sample

2.11 Curve Sketching (part 1) using limits and IVT

I) determine the domain of f(x)

II) determine symmetries of f(x) (even or odd function)

III) determine where f(x) changes sign, record points

IV) find points of discontinuity

V) Determine type of discontinuity (denote appropriately)

VI) draw v - asymptotes and holes for RDs

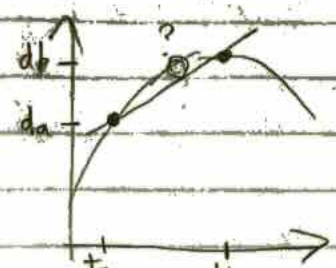
VII) Find horizontal asymptotes by evaluating limits to ∞

VIII) Construct a sketch of the graph that is as accurate as possible

CHAPTER 3: DERIVATIVES (!)

3.1 Instantaneous velocity

How do we find the velocity of an object at a specific moment in time? We can't use $\Delta d / \Delta t$ if $\Delta t = 0$.



One approach is drawing a secant line between two points on either side of the one we wish to measure. This gives us the average velocity over that interval:

$$V_{ave} = \frac{d_b - d_a}{t_b - t_a} = \frac{\Delta d}{\Delta t}$$

To get the actual (and not approximate), Δt must approach 0. So, if $d(t)$ is displacement as a function of time, we have

$$v(t) = \lim_{\Delta t \to 0} \frac{d(t + \Delta t) - d(t)}{\Delta t} = d'(t)$$

3.2 Definition of a derivative

Our velocity example can apply to any function. For $f(x)$, the approximate rate of change at $x = a$ is

$$\left.\frac{f(a+h) - f(a)}{h}\right\} \text{Newtonian Quotient}$$

Thus, the actual rate of change at $x = a$ is defined by

$$f'(a) = \lim_{h \to 0} \left.\frac{f(a+h) - f(a)}{h}\right\} \text{derivative at } x = a$$

It can also be defined as $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

3.2.1 The tangent line

If a line is tangent to a curve at $x = a$, its slope m is $f'(a)$ and it passes through $(a, f(a))$ assuming x is differentiable at a. The full equation is given by

$$y = f'(a)(x-a) + f(a)$$

3.2.2 Differentiability (and continuity)

If $f'(a)$ $\lim_{t \to a} \frac{f(t) - f(a)}{t - a}$ exists and is equal to $f'(a)$.

Since the denominator approaches 0 as $a \to t$, we must have $f(t) - f(a) \to 0$ as $t \to a$ if $f(x)$ is differentiable at a. This is the definition for continuity, so differentiability implies continuity

However, continuity does not imply differentiability (ex: $|x|$ is continuous over $\mathbb{R}$, but un-differentiable at $x = 0$. [sharp corner on graph]

Intuition: if $f'(x)$ is differentiable, $f'(x)$ is continuous [over some interval I]

3.3 The derivative function

$f(x)$ is differentiable on I if $f'(a)$ exists for each $a \in I$. In that case, we have:

$$f'(x) = \frac{dy}{dx} = y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{d}{dx} f(x)$$

We can take the higher order derivative by taking a derivative of a derivative of a...

ex. second derivative: $f''(x) = \frac{d^2y}{dx^2} f(x)$

ex. nth derivative: $f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d}{dx} f^{(n-1)}(x)$

3.4 Derivatives of elementary functions

Constant function : $f(x) = n \Rightarrow f'(x) = 0$

$\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \frac{f(n) - f(n)}{h} = 0$ {fuzzy logic}

Linear function : $f(x) = mx + b \Rightarrow f'(x) = m$

$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{m(a+h) - b - ma - b}{h} = \lim_{h \to 0} \frac{mh}{h} = m$

Quadratic function (simple) : $f(x) = x^2 \Rightarrow f'(x) = 2x$

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} 2x + h = 2x$

Trig functions (simple) : $\frac{d}{dx} \sin(x) = \cos(x)$ and $\frac{d}{dx} \cos(x) = -\sin(x)$

[derived using fundamental trig limit]

e^x : $f(x) = e^x \Rightarrow f'(x) = e^x$ (wow)!

for exponential functions : $\lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h} =$

$= a^x \lim_{h \to 0} \frac{a^h - 1}{h} = a^x \cdot f'(0)$. So, if we want $f'(x) = f(x)$,

we need $f'(0) = 1$. So, $1 = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^h - e^0}{h}$

$= \lim_{h \to 0} \frac{e^h - 1}{h}$. Plugging this in gives $\lim_{h \to 0} a^x \cdot 1 = a^x$. So, $\frac{d}{dx} e^x = e^x$

Exponential function : $\frac{d}{dx} a^x = a^x \cdot \log_e(a) = a^x \cdot \ln(a)$ } derived from e^x derivation

3.5 Tangent Lines + Linear Approximation

A tangent line at $x=a$ approximates that function near $x=a$

$$\left. f(x) \sim f'(x)(x-a) + f(a) = L_a^f(x) \quad \right] \text{near } x=a$$

This works if x is differentiable at a.

$L_a^f(x)$ (linear approximation of x) has 3 important properties

I) $L_a^f(a) = f(a)$ II) $L_a^f(x)$ is differentiable

III) $L_a^f(x)$ is the only linear function with these properties

The error of $L_a^f(x)$ is equal to:

$$\text{Error} = |f(x) - L_a^f(x)|$$

Note 1: we may have $L_a^f(x) = f(x)$ when x is far from a. However, examples of this are trivial. We say the further x is from a, the larger potential for error there is.

Note 2: The more curved a function is at $x=a$, the faster the error grows when x moves away from a. So, the larger $|f''(a)|$, the faster the error bound grows.

Thus, we can approximate error using $f''(x)$:

$$|f(x) - L_a^f(x)| \leq \frac{M}{2}(x-a)^2 \text{ over } I \text{ if } \forall x \in I,\ |f''(x)| \leq M$$

3.7 Arithmetic Rules for differentiation

The sum rule: $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

The product rule: $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x)$

$\hookrightarrow$ The constant function rule: $\frac{d}{dx}c \cdot f(x) = c \cdot \frac{d}{dx}f(x)$

The reciprocal rule:

Let $h(x) = \frac{1}{g(x)}$. $h'(x) = -\frac{g'(x)}{g^2(x)}$

The quotient rule: $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

The power rule: $\frac{d}{dx}x^{\alpha} = \alpha x^{\alpha-1}$ $\forall \alpha \in \mathbb{R} \cdot x^{\alpha-1}$ is defined

-> because of this, polynomial functions are very easy to differentiate

3.8 The chain rule

Chain rule: $\frac{d}{dx}g(f(x)) = g'(f(x)) \cdot f'(x)$ $\boxed{OR}$ $\frac{dv}{dx} = \frac{dy}{dx} \cdot \frac{dv}{dy}$

Proof. Let $L_a^f(x)$ be the linear approximation of $f(x)$ at $x = a$, and let $L_{f(a)}^g(y)$ be the linear approximation of $g(y)$ at $y = f(a)$. So, the linear approximation/tangent line to $g(f(x))$ is

$$L_{f(a)}^g(L_a^f(x))$$
$$= g(f(a)) + g'(f(a))(L_a^f(x) - f(a))$$
$$= g(f(a)) + g'(f(a))(f(a) + f'(a)(x-a)) - f(a))$$
Sub in linear approx
$$= \underbrace{g(f(a))}_{h(a)} + \underbrace{g'(f(a))(f'(a))}_{h'(a)}\underbrace{(x-a)}_{x-a} = \text{tangent line of } h(x)$$

We can show that this is the tangent line to $g(f(x))$ at $x = a$. Thus,

$\frac{d}{dx}g(f(x)) = g'(f(x))f'(x)$.

3.9 More trig identities

$$\boxed{\frac{d}{dx}\tan(x) = \sec^2(x)}$$

Proof: $\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)} = \frac{(\frac{d}{dx}\sin(x))(\cos(x)) - (\frac{d}{dx}\cos(x))(\sin(x))}{\cos^2(x)} =$

$\frac{(\cos(x))(\cos(x)) - (-\sin(x))(\sin(x))}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$

$$\boxed{\frac{d}{dx}\sec(x) = \tan(x)\sec(x)}$$

proof: $\frac{d}{dx}\sec(x) = \frac{d}{dx}\frac{1}{\cos(x)} = \frac{0\cdot\cos(x) - 1\cdot(-\sin(x))}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)} =$

$= \frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\cos(x)} = \tan(x)\sec(x)$

$\boxed{\frac{d}{dx}\csc(x) = -\cot(x)\csc(x)}$ (similar proof to $\sec(x)$)

3.10 Derivatives of Inverse Functions

IFT: If $y = f(x)$ is continuous and invertable on $[c,d]$ with the inverse defined as $x = g(y)$) and $f(x)$ is differentiable at $a \in (c,d)$, then if $f'(a) \neq 0$ and $g(y)$ is differentiable at $b = f(a)$, we have

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(g(b))} = g'(f(a))$$

Moreover, L_a^f is also invertable:

$$[L_a^f]^{-1}(x) = L_b^g(x) = L_{f(a)}^g(x)$$

Chain rule proof: Let $g(y)$ be the inverse of $f(x)$. As such

$$g(f(x)) = x \Rightarrow \frac{d}{dx}g(f(x)) = \frac{d}{dx}x = 1 \Rightarrow g'(f(x))f'(x) = 1$$

$$\Rightarrow g'(f(x)) = \frac{1}{f(x)} \Rightarrow f'(x) = \frac{1}{g'(f(x))}$$

3.11 Derivatives of Inverse Trigonometric Functions

$$\boxed{\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}}$$

proof: $\sin(\sin^{-1}(x)) = x \Rightarrow \frac{d}{dx}\sin(\sin^{-1}(x)) = \frac{d}{dx}x = 1 \Rightarrow$

$\cos(\sin^{-1}(x)) \cdot \frac{d}{dx}\sin^{-1}(x) = 1 \Rightarrow \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))} \Rightarrow$

since $\cos(y) = \sqrt{1-\sin^2(y)}$, $\Rightarrow \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-\sin^2(\sin^{-1}(x))}} \Rightarrow$

$\Rightarrow \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

$$\boxed{\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}}$$ (similar proof)

$$\boxed{\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}}$$

proof: $\tan(\tan^{-1}(x)) = x \Rightarrow \frac{d}{dx}\tan(\tan^{-1}(x)) = \frac{d}{dx}x \Rightarrow$

$\Rightarrow \sec^2(\tan^{-1}(x)) \cdot \frac{d}{dx}\tan^{-1}(x) = 1 \Rightarrow \frac{d}{dx}\tan^{-1}(x) = \frac{1}{\sec^2(\tan^{-1}(x))} \Rightarrow$

since $\sec^2(y) = 1+\tan^2(y)$, $\Rightarrow \frac{1}{1+\tan^2(\tan^{-1}(x))} = \frac{d}{dx}\tan^{-1}(x) \Rightarrow$

$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

3.12 Implicit Differentiation

We can evaluate the derivative of implicit functions by taking the derivative of both sides:

uses chain and product rule

$$x^3 + y^3 = 6xy \Rightarrow \frac{d}{dx}(x^3+y^3) = \frac{d}{dx}6xy \Rightarrow \frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}6xy$$

$$\Rightarrow 3x^2 + \frac{dy}{dx}(3y^2) = \frac{dy}{dx}6x + 6y \Rightarrow 3x^2 - 6y = \frac{dy}{dx}6x - \frac{dy}{dx}3y^2 \Rightarrow$$

$$3x^2 - 6y = \frac{dy}{dx}(6x - 3y^2) \Rightarrow \frac{dy}{dx} = \frac{3x^2-6y}{6x-3y^2}$$

However, must make sure the function actually has solutions, or we will have found a derivative that doesn't apply to any points

Logarithmic Differentiation

We can use implicit differentiation for functions of the form $f(x)^{g(x)}$:

Let $f(x) = x^x$. Determine $f'(x)$.

$y = x^x \Rightarrow \ln(y) = \ln(x^x) = x\ln(x)$. Taking the derivative, we h[ave]

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}x\ln(x) \Rightarrow \frac{dy}{dx}\frac{1}{y} = \ln(x) + x\frac{1}{x} = \ln(x) + \frac{x}{x} = \ln(x)+1$$

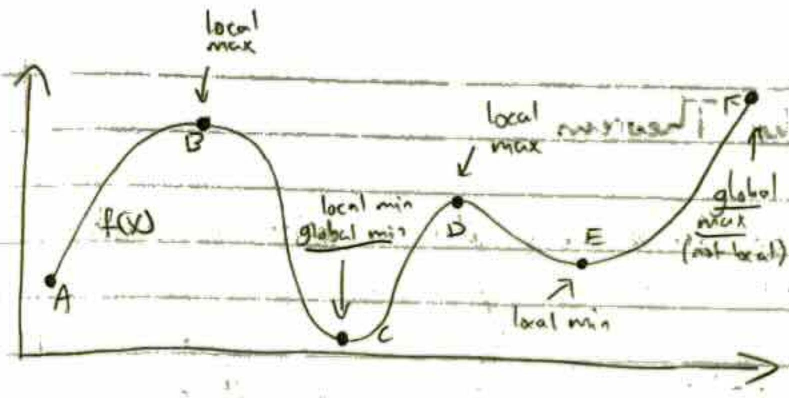
$\Rightarrow \frac{dy}{dx} = y(\ln(x)+1)$. Since $y = x^x$, we have $\frac{dy}{dx} = x^x(\ln(x)+1)$

3.13 Local Extrema

A point c is a local maximum of a function f if there exists an open interval (a,b) containing c such that $f(x) \le f(c)$ for all $x \in (a,b)$

→ Local min: $f(x) \ge f(c)$ for all $x \in (a,b)$

Remember, a global max/min is over a large interval I (which is closed



point C is the lowest value of the function, so it is both a global and local minimum

F is not defined on both sides, so it cannot be a local max (although it is the largest value of F)

3.13.1 Local Extrema Theorem

If c is a local maximum or minimum for f and $f'(c)$ exists, it must be that $f'(c) = 0$

However, the converse is not true: there are points c where $f'(c) = 0$ but c is neither a local minimum or maximum
-> Ex: $x = 0$ for $f(x) = x^3$

In fact, $f'(c)$ doesn't even need to be defined for c to be an extremum
-> Ex $x = 0$, $f(x) = |x|$

As such, if $f'(c) = 0$ or $f'(c)$ doesn't exist (and c is in the domain of f), we define it as a critical point

3.14 : Related Rates

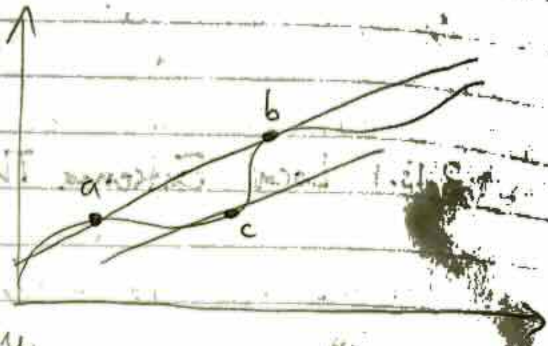
Using rates of change to model real situations

Strategies
- -> List all values that are available (both constants and rates)
  - -> can the chain rule be applied?
- -> Model the situation with an equation, take the derivatives of this equation with respect to time
  - -> Pythagorean theorem comes up often

4.1: The Mean Value Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Geometrically, there is some point on $[a,b]$ where the slope of f is the same as the slope of the a-b secant line

Example: if a car travels with an average speed (secant slope) over a given interval, the car must be travelling at exactly that speed at some point on that interval

Proof: three cases (when $a = b$) (called Rolle's theorem)

- -> constant function 0: obvious $f'(c) = 0$ exists
- -> otherwise, by EVT, there must be a max and min over $[a,b]$, so $f'(c) = 0$ (and $a = b$, so secant line is also 0)

Rolles theorem: subset of MVT: if $f(a), f(b) = 0$, there is a value $a < c < b$ such that $f'(c) = 0$ (there is a critical point somewhere, and that critical point is a max or min)

4.2: Applications of MVT

4.2.1 Antiderivatives

For some function $f(x)$, its antiderivative $F(x)$ is the function (or group of functions) such that $F'(x) = f(x)$. Since derivatives of constant terms are 0, $\frac{d}{dx}(F(x) + c)$ is also $F(x)$ for any c

-> Ex. If $f(x) = x^2$, $F(x) = \frac{1}{3}x^3 = \frac{1}{3}x^3 + 1 = \frac{1}{3}x^3 + 2$ ..

This can be used to prove the constant functions theorem

Assume that $f'(x) = 0$ for all $x \in I$. There exists an α such that $f(x) = \alpha$ for every $x \in I$

This leads to the antiderivative theorem

Assume $f'(x) = g'(x)$ for all $x \in I$. Then there exists an α such that $f(x) = g(x) + \alpha$ for every $x \in I$

We denote the antiderivative of $f(x)$ as $\int f(x)\, dx$
-> aka indefinite integral of $f(x)$

Power rule for antiderivatives: if $\alpha \neq -1$, $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C$
since the derivative of this expression is x^α

Like derivation, $\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$

$\int \frac{1}{x} dx = \ln|x| + C$ in any interval not containing $x = 0$.

$\int e^x dx = e^x + C$ $\qquad$ $\int a^x dx = \frac{a^x}{\ln(a)} + C$

$\int \sin(x)\, dx = -\cos(x)$ $\qquad$ $\int \cos(x)\, dx = \sin(x)$

$\int \sec^2(x)\, dx = \tan(x) + C$ $\qquad$ $\int \frac{1}{x^2+1} dx = \arctan(x) + C$

$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$ $\qquad$ $\int \frac{1}{\sqrt{1-x^2}} dx = \arccos(x) + C$

Increasing Function Theorem 4.2.2

Let f be defined on the interval I
- -> f is increasing on I $\Leftrightarrow$ $f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$, $x_1 < x_2$
- -> f is decreasing on I $\Leftrightarrow$ $f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$, $x_1 < x_2$
- -> f is non-decreasing / non-increasing : negation of $f(x_1) </> f(x_2)$

Increasing / Decreasing Function Theorem:
- -> let I be an interval and x_1, x_2 be points on I

If $f'(x) > 0$ on I, f is increasing on I (by x_1, x_2 definition)

If $f'(x) < 0$ on I, f is decreasing

If $f'(x) \geq 0$ on I, f is non-decreasing

If $f'(x) \leq 0$ on I, then f is non-increasing

4.2.3 Functions with Bounded Derivatives

Assume f is continuous on $[a,b]$ and differentiable on (a,b). Also denote the maximum value of $f'(x)$ over (a,b) as M and the minimum as m. As such, for all $x \in (a,b)$, $m \leq f'(x) \leq M$. By the bounded derivative theorem, we have

$$f(a) + m(x-a) \leq f(x) \leq f(a) + M(x-a) \quad \text{for all } x \in [a,b]$$

Basically, the possible growth of $f(x)$ on (a,b) is bounded by the largest value of its derivative on the top (M) and smallest value of its derivative on the bottom. (m).

4.2.4 Comparing Functions Using their Derivatives

Let f, g be continuous and differentiable for $x \geq a$ and let $f(a) = g(a)$

→ If $f'(x) \leq g'(x)$ for all $x > a$, then $f(x) \leq g(x)$ for all $x > a$

→ If $f'(x) \leq g'(x)$ for all $x < a$ then $f(x) \geq g(x)$ for all $x < a$

This can be used to show $e^{\alpha} = \lim_{n \to \infty} \left(1 + \frac{\alpha}{n}\right)^n$

4.2.5 Interpreting the Second derivative

Since $f''(x) = \frac{d}{dx} f'(x)$, $f''(x)$ represents the rate of change of the rate of change of $f(x)$

→ how quickly the tangent slope is changing

If $f''(x) > 0$, the function is curving (concave) upwards

If $f''(x) < 0$, the function is curving (concave) downwards

Formal Definition of Concavity

If f is concave upwards over I, for every pair of points a, b in I, the secant line joining $(a, f(a))$ and $(b, f(b))$ lies above the graph of f.

→ for downwards: same, but below

If $f''(x) > 0$ for each $x \in I$, the ~~the~~ graph of f is concave upwards on I

If $f''(x) < 0$ for each $x \in I$, the graph of f is concave downwards on I

A point $(c, f(c))$ is an inflection point for the function f if

1) f is continuous at $x = c$
2) the concavity of f changes at $x = c$

If f'' is continuous at c and $(c, f(c))$ is an inflection point, then $f''(c) = 0$.

However, the converse isn't true: if $f''(c) = 0$, $(c, f(c))$ isn't necessarily an inflection point
-> ex. $f(x) = x^4$, $c = 0$ ($f''(c) = 0$ but concavity doesn't change)

Steps for finding Inflection Points
1) Find $f''(x)$
2) Find all c such that $f''(c) = 0$ or DNE. These are the candidates for inflection points
3) For each c, check if
   a) f is continuous at c
   b) the concavity changes at c

   if these are true, $(c, f(c))$ is an inflection point

4.2.7: Classifying Critical Points

First derivative test
-> Assume c is a critical point of f and f is continuous at c
-> if there is an interval (a, b) containing c such that
   -> $f'(x) < 0$ for all $x \in (a, c)$
   -> $f'(x) > 0$ for all $x \in (c, b)$

   then f has a local minimum at c
-> reverse signs / inequalities for a local maximum at c

Strategy: find each critical point. Calculate the sign of the derivative on the interval between each critical point (there will only be one sign by definition). Apply first derivative test as needed.

Second derivative test

→ Assume $f'(c) = 0$ and f'' is continuous at $x = c$

→ If $f''(c) < 0$, then f has a local maximum at c

→ If $f''(c) > 0$, then f has a local minimum at c

This is easier to compute (since only one critical point need be calculated), but requires f'' to be continuous at $x = c$

Finding Global Maxima or Minima on $[a, b]$

1) Evaluate $f(a)$ and $f(b)$
2) Find all critical points on (a, b)
3) Evaluate the function at each of these points
4) The largest of all of these values is the global max, the smallest is the global min

4.3 L'Hôpital's Rule

Motivation: let $\lim_{x \to a} f(x) = \lim_{x \to a} = 0$, meaning $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$. We know that for x near a, $h(x) \approx h(a) + h'(a)(x-a)$. Plugging that into our limit, we have

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)} = \frac{f'(a)}{g'(a)} \quad (\text{since } f(a), g(a) = 0).$$

This leads us to hôpital's rule:

Assume $f'(x)$ and $g'(x)$ exist near $x = a$, $g'(x) \neq 0$ near $x = a$ (except possibly at a) and that $\lim_{x \to a} \frac{f(x)}{g(x)}$ is either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (indeterminate form) then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad (\text{provided this limit exists})$$

4.4 : Curve Sketching Part 2

Strategy for curve sketching : part 2

1) Determine the domain of $f(x)$
2) Determine the x-intercept(s) of $f(x)$ ($f(x) = 0$)
3) Determine the y-intercept of $f(x)$ ($f(0)$)

4) Determine the vertical asymptotes of $f(x)$ (if applicable)
5) Determine the horizontal asymptotes of $f(x)$ by evaluating $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

6) Calculate the derivative of $f(x)$
   a) determine critical points ($f'(x) = 0$ and $f'(x)$ DNE)

7) Calculate second derivative of $f(x)$
   d) determine points where $f''(x) = 0$ or DNE

8) Make table with all important points and intervals between those points as the header and f, f', f'' as the column. For each interval, determine if f' and f'' are positive or negative, then list f as increasing/decreasing and concave up/down

9) Determine the intervals where f is increasing, decreasing, concave up or concave down

10) Finally sketch the damned thing

5.1 Intro to Taylor Polynomials and Approximation

Recall that if f is differentiable at $x = a$, then if $x \cong a$

$$f'(a) \doteq \frac{f(x) - f(a)}{x - a} \Rightarrow f'(a)(x-a) \doteq f(x) - f(a) \Rightarrow$$

$$f(x) \doteq L_a^f(x) = f(a) + f'(a)(x-a)$$

This is the linear ~~approximation given by the~~ [illegible]

Remember, $L_a^f(a) = f(a)$ and $L_a^{f\,\prime}(x) = f'(x)$

-> Encodes the value of the function and its derivative at a

The error is of course calculated $\text{Error}(x) = |f(x) - L_a^f(x)|$ and is 0 at $x = a$. Since the 0^{th} and 1^{st} derivatives are accounted for, the size of the second derivative determines the possible error size

If we wanted to account for this error, we could add a term to the linear approximation to get

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

from error bound ??

This encodes not only 0^{th} and 1^{st} derivative information, but also 2^{nd} derivative information as well. This is the second degree Taylor polynomial centered at a, denoted $T_{2,a}(x)$

This is better at approximating $f(x)$ than $T_{1,a}(x) = L_a(x)$. However, we can keep adding approximations:

$$T_{3,a} = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

This pattern comes from solving $p(x) = c_0 + c_1(x-a) + c_2(x-a)^2$ -> $c_0 = f(a)$ (since c_0 is the only term that isn't 0 when $x = a$), then solving $p'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 \ldots$, etc (keep checking for the value at $x = a$)

The full taylor polynomial [illegible] ?

$$T_{n,a} = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$ as long as $f(x)$ is n-times differentiable

$T_{n,a}$ encodes information about the first n derivatives : $T_{n,a}^{(k)}(a) = f^{(k)}(a)$

Errors in approximation and Taylor's theorem

If f is n-times differentiable at $x=a$, then

$$R_{n,a}(x) = f(x) - T_{n,a}(x)$$

Where $R_{n,a}(x)$ is the n-th degree Taylor remainder function centered at $x=a$

Unsurprisingly, we have $\text{Error}(x) = |R_{n,a}(x)|$

So, how can we estimate the size of $R_{n,a}(x)$ (and our error)?

Taylor's theorem: Assume f is $n+1$ times differentiable on the interval I containing $x=a$. Let $x \in I$. Then, there exists a point c such that c is between x and a and

$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Intuitively, this means that the error in our approximation of a function centered at a ($R_{n,a}(x)$) is equal to

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$ where c is some between x and a (here x is wherever we are taking our approximation). If we know the behaviour of $f^{(n+1)}(x)$, we can get an idea for how big our error might be (since we know a and x, and know c must be between them)

$\frac{\sin(1)}{2} = \frac{0}{1} = 0$

Three observations about Taylor's theorem:

1) For $n=1$, we have $R_{1,a}(x) = \frac{f''(c)}{2}(x-a)^2$, which we saw before. This tells us that the error in linear approximation depends on the size of $f''(x)$ over $[x,a]$, and the difference between x and a ($|x-a|$)

2) When $n=0$, we have $R_{0,a}(x) = f(x) - T_{0,a}(x) = \frac{f'(c)}{1!}(x-a)^1$

Since $T_{0,a}(x) = f(a)$, we have $f(x) - f(a) = f'(c)(x-a)$
$\Rightarrow \frac{f(x)-f(a)}{x-a} = f'(c)$ for some c between x and a.

This is the MVT; Taylor's theorem is a higher-order version of the MVT

3) Since Taylor's theorem doesn't tell us what c is or how to find it, it is only useful if we have some idea what $f^{(n+1)}(x)$ looks like for all x. Luckily, there are many functions where this is the case:
$\rightarrow \sin(x)$, $\cos(x)$,
$\rightarrow e^x$

Basically, functions where derivatives repeat after a certain order

Using Taylor's Theorem to calculate limits

Taylor's theorem can be used to find limits in the form

$$\lim_{x\to a} \frac{f(x) - T_{n,a}(x)}{\text{something}}, \text{ usually } \lim_{x\to 0} \frac{f(x) - T_{n,0}(x)}{\text{something}}$$

Example: Find $\lim_{x\to 0} \frac{\sin(x) - x}{x^2}$ w/o hôpital's rule

Since $f(x) = \sin(x)$, $T_{2,0}(x) = x$. So, by Taylor's theorem, we have

$|\sin(x) - x| = \left|\frac{-\cos(c)}{3!}x^3\right|$ for some c between x and 0

Since $|-\cos(c)| \le 1$ for all c, we have

$$|\sin(x) - x| = \left|-\frac{\cos(c)}{3!}x^3\right| \le \frac{1}{3!}|x^3| = \frac{1}{6}|x|^3$$

This means, using properties of absolute values:

$-\frac{1}{6}|x|^3 \le \sin(x) - x \le \frac{1}{6}|x|^3$. We divide by x^2 to get

$$-\frac{1}{6x^2}|x|^3 \le \frac{\sin(x) - x}{x^2} \le \frac{1}{6x^2}|x|^3 \Rightarrow -\frac{|x|}{6} \le \frac{\sin(x) - x}{x^2} \le \frac{|x|}{6}$$

Since $\lim_{x\to 0} -\frac{|x|}{6} = \lim_{x\to 0} \frac{|x|}{6} = 0$, by the squeeze theorem, we have

$\lim_{x\to 0} \frac{\sin(x) - x}{x^2} = 0$, which is confirmed by hôpital's rule

Taylor's approximation theorem I

If $f^{(k+1)}$ is continuous on $[-1, 1]$, then so built $g(x) = \frac{f^{(k+1)}(x)}{(k+1)!}$

By the EVT, $g(x)$ has a maximum on $[-1, 1]$, so there exists some M such that $\left|\frac{f^{(k+1)}(x)}{(k+1)!}\right| \le M$ for all $x \in [-1, 1]$

Let $x \in [-1, 1]$. Taylor's theorem assures us that there exists a c between x and 0 such that:

$$R_{k,0}(x) = \left|\frac{f^{(k+1)}(c)}{(k+1)!}(x)^{k+1}\right| \Rightarrow |f(x) - T_{k,0}(x)| = \left|\frac{f^{(k+1)}(c)}{(k+1)!}x^{k+1}\right| \le M|x|^{k+1}$$

As such, $-M|x|^{k+1} \le f(x) - T_{k,0}(x) \le M|x|^{k+1}$

This is Taylor's theorem:

Assume $f^{(k+1)}$ is continuous on $[-d, d]$ for $d > 0$. Then, there exists a constant $M > 0$ such that

$$|f(x) - T_{k,0}(x)| \leq M|x|^{k+1}$$, and as such,

$$-M|x|^{k+1} \leq f(x) - T_{k,0}(x) \leq M|x|^{k+1}$$

This is useful for calculating limits: To do this, you must have $x \to 0$ and use the squeeze theorem:
→ show $-M|0|^{k+1} \leq$ your limit $\leq M|0|^{k+1}$ since $-M|0|^{k+1} = M|0|^{k+1} = 0$

Big-O Notation

How can we formalize that some functions approach a value faster than others (ex x^2 approaches 0 faster than x)?

We say f is big-O of g ($f(x) = O(g(x))$) if as $x \to a$ there exists some $\varepsilon > 0$ and $M > 0$ such that $|f(x)| \leq M|g(x)|$ for all $x \in (a-\varepsilon, a+\varepsilon)$ except possibly at $x = a$

We write $f(x) = O(g(x))$ as $x \to a$, or simply $f(x) = O(g(x))$ if a is understood

Intuitively, the order of magnitude of $g(x)$ is the same as $f(x)$ (or rather, less than or equal to that of $f(x)$)

Theorem: For any $n \in \mathbb{N}$, $f(x) = O(x^n) \Rightarrow \lim_{x \to 0} f(x) = 0$
→ Proof: using $-M|x^n| \leq f(x) \leq M|x^n|$ and squeeze theorem ($x = 0$)

This can be summarized as: $\lim_{x \to 0} O(x^n) = 0$

Extended Big-O Notation

Suppose f, g, and h are defined on an open interval containing $x=a$, except for possibly $x=a$. We have

$$f(x) = g(x) + O(h(x)) \text{ as } x \to a \quad \text{if}$$
$$f(x) - g(x) = O(h(x)) \text{ as } x \to a$$

This tells us that near $x=a$, the order of magnitude of the error $|f(x) - g(x)|$ is at most that of $h(x)$

For example, consider $f(x) = \sin(x)$ on $[-1,1]$. By Taylor's theorem,

$$|\sin(x) - T_{1,0}(x)| = |\sin(x) - x| = \left|\frac{f''(c)}{2!}x^2\right| = \left|\frac{\sin(c)}{2}x^2\right| \leq \frac{1}{2}x^2$$

As such, $\sin(x) - x = O(x^2)$, so $\sin(x) = x + O(x^2)$. We can also get $\sin(x) = x + O(x^3)$ since $T_{2,0}(x)$ is also x for $f(x)$

Taylor's approximation theorem 2

Let $r>0$. If f is $(n+1)$-times differentiable on $[-r,r]$ and $f^{(n+1)}$ is continuous on $[-r,r]$, then $f(x) = T_{n,0}(x) + O(x^{n+1})$, $x \to 0$

Let $f(x) = O(x^n)$ and $g(x) = O(x^m)$ as $x \to 0$. What is big O of $h(x) = f(x) + g(x)$?

Since $\lim_{x\to0} f(x) = \lim_{x\to0} g(x) = 0$, we have $\lim_{x\to0} f(x) + g(x) = 0$. Using the triangle inequality, we can show that $O(x^n) + O(x^m) = O(x^k)$ where $k = \min\{n,m\}$. In other words, the error of a sum is at least as large as the error of either part.

Big-O Arithmetic

Assume that $f(x) = O(x^n)$ and $g(x) = O(x^m)$ as $x \to 0$ for some $m, n \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then we have

1) $c \cdot (O(x^n)) = O(x^n)$ (constants don't impact big-O)

2) $O(x^n) + O(x^m) = O(x^k)$ where $k = \min\{n, m\}$

3) $O(x^n) \cdot O(x^m) = O(x^{n+m})$

4) If $k \leq n$, $f(x) = O(x^n) = O(x^k)$

5) If $k \leq n$, then $\frac{1}{x^k} O(x^n) = O(x^{n-k})$. In other words, $f(x)/x^k = O(x^{n-k})$

6) $f(u^k) = O(u^{kn})$, meaning we can substitute $x = u^k$

Calculating Taylor Polynomials

It turns out the converse to Taylor's approximation theorem II is true:

Assume $r > 0$, and that f is $(n+1)$ times differentiable on $[-r, r]$ and $f^{(n+1)}$ is continuous on $[-r, r]$. If p is a polynomial of degree n or less with $f(x) = p(x) + O(x^{n+1})$, then

$p(x) = T_{n,0}(x)$.

This can be verified by big-O arithmetic.

This makes it easy to calculate Taylor Polynomials of high degrees without calculating each term